Homework 3. January 26, 2017.

1. (Equivariant Darboux) Suppose G is a compact Lie group that acts on the symplectic manifold (M, ω), and the action preserves ω. Suppose $p \in M$ is a fixed point of the action. Show that there is a chart in a neighbourhood of p, denoted by $\phi : U \to \mathbb{R}^{2n}$ such that

   (a) U is G-invariant and $\phi(p) = 0$. There is a linear action of G on $\mathbb{R}^{2n}$ under which ϕ is equivariant, i.e. $\phi(gu) = g\phi(u)$ for all $g \in G$ and $u \in U$,

   (b) and $\phi^* \omega_{std} = \omega$.

   (Hint: You need a G-invariant Riemannian metric in the proof of the relative version of Moser theorem.)

2. Suppose V is a vector space with a skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. Show that V has a basis $e_1, \ldots, e_k, f_1, \ldots, f_k, g_1, \ldots, g_l$ such that $\Omega = \sum_{i=1}^{k} e_i^* \wedge f_i^*$ and $\ker \omega = \langle g_1, \ldots, g_l \rangle$.

3. Suppose X is a smooth manifold and ω_{can} is the canonical symplectic form on T^*X. Along the zero section of T^*X, the tangent space splits as $T(T^*X)|_X = TX \oplus T^*X$. Show that the symplectic form at $(x, 0)$ is

$$\omega_{can}((v_0, f_0), (v_1, f_1)) = f_0(v_1) - f_1(v_0), \quad (v_i, f_i) \in T_{(x,0)}T^*X = T_xX \oplus T_x^*X.$$