Homework 2. January 16, 2017.

1. Suppose $X \subset M$ is a submanifold. The co-normal bundle N^*S over S is the sub-bundle of $T^*M|_S$ that annihilates the tangent bundle TS. Show that the co-normal bundle is a Lagrangian submanifold of T^*M.

2. Suppose (M_1, ω_1), (M_2, ω_2) are symplectic manifolds. Define a two-form $\tilde{\omega}$ on the product $M_1 \times M_2$ as

$$\tilde{\omega} := \mathrm{pr}_1^* \omega_1 - \mathrm{pr}_2^* \omega_2,$$

where $\mathrm{pr}_i : M_1 \times M_2 \to M_i$ is a projection map for $i = 1, 2$. Show that

   (a) $\tilde{\omega}$ is a symplectic form.

   (b) A diffeommorphism $\phi : M_1 \to M_2$ is a symplectomorphism if and only if its graph $\Gamma_\phi \subset (M_1 \times M_2, \tilde{\omega})$ is a Lagrangian submanifold.

3. Suppose M is a compact Riemannian manifold, and let $\langle \cdot, \cdot \rangle : \Omega^*(M) \times \Omega^*(M) \to \mathbb{R}$ be the induced inner product on forms. Define the operator $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ as $d^* := (-1)^{n(k+1)+1} * d *$, where $*$ is the Hodge star. Show that $\langle d\alpha, \beta \rangle = \langle \alpha, \mathrm{d}^* \beta \rangle$ for all forms $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$.