BLOW-UPS ON SYMPLECTIC MANIFOLDS

1. View 1

The blow-up of $\mathbb{C}^n$ at the origin is the complex manifold

$$\mathrm{Bl}_0\,\mathbb{C}^n := \{(z,w) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \forall i,j\}.$$

This space is obtained by deleting the origin in $\mathbb{C}^n$ and putting a $\mathbb{CP}^{n-1}$ in its place. $\mathbb{CP}^{n-1}$ is the set of complex lines through the origin.

The blow-up comes with two projection maps

$$\pi : \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{C}^n, \quad pr : \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{CP}^{n-1}. \tag{1}$$

Away from the set $\pi^{-1}(0)$, π is a biholomorphism onto $\mathbb{C}^n \backslash \{0\}$. The inverse image $\pi^{-1}(0)$ is $\mathbb{CP}^{n-1}$, this is called the *exceptional divisor* corresponding to the blow-up. The fibers of the second map in (1) are complex lines. So, $\mathrm{Bl}_0\,\mathbb{C}^n$ is a line bundle over $\mathbb{CP}^{n-1}$. In fact it is the universal line bundle – the fiber above $w \in \mathbb{CP}^{n-1}$ is the complex line in $\mathbb{C}^n$ represented by w. The exceptional divisor is the zero section of the line bundle.

In the previous lecture, we showed that for any $\lambda > 0$,

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 pr^* \omega_{FS} \tag{2}$$

is a Kähler form on $\mathrm{Bl}_0\,\mathbb{C}^n$. Observe that ω_λ restricts to the form $\lambda^2 \omega_{FS}$ on the exceptional divisor. View 1 of $\mathrm{Bl}_0\,\mathbb{C}^n$ sees it as a complex manifold with a Kähler form.

2. View 2

Lemma 2.1. *Suppose $\lambda > 0$. For any $\delta \in (0, \infty]$, the map*

$$\begin{aligned} F : (\pi^{-1}(B_\delta \backslash \{0\}), \omega_\lambda) &\to (B_{\sqrt{\delta^2+\lambda^2}} \backslash B_\lambda, \omega_{std}) \\ z &\mapsto Z := \sqrt{|z|^2 + \lambda^2}\,\frac{z}{|z|} \end{aligned}$$

is a symplectomorphism.

Proof. Recall that

$$\begin{aligned} \omega_\lambda &= \frac{\iota}{2} \partial \overline{\partial} (\lambda^2 \log |z|^2 + |z|^2) \\ &= \frac{\iota}{2} (dz \wedge \mathrm{d}\overline{z} + \frac{\lambda^2 dz \wedge d\overline{z}}{|z|^2} - \frac{\lambda^2 \overline{z}.dz \wedge z.d\overline{z}}{|z|^4})) \end{aligned}$$

Plug in the expression for Z in $F^* \omega_{std} = dZ \wedge d\overline{Z}$ to finish the proof. □

This Lemma shows that $\mathbb{C}^n \backslash \overline{B}_\lambda$ with the standard symplectic structure is symplectomorphic to $\mathrm{Bl}_0\,\mathbb{C}^n$ minus the zero section. This suggests that as a symplectic manifold, $\mathrm{Bl}_0\,\mathbb{C}^n$ can be viewed as $(\mathbb{C}^n \backslash B_\lambda)/\sim$. Here, the equivalence relation $\sim$ is given by the S^1-action on the boundary ∂B_λ. This is VIEW 2 of $\mathrm{Bl}_0\,\mathbb{C}^n$.

Remark 2.2. The space $\mathbb{C}^n \backslash \overline{B}_\lambda$ does not have boundary – the subspace $\partial B_\lambda / S^1$ is a symplectic submanifold of dimension $2(n-1)$.

Remark 2.3. (Equivalence of View 1 and View 2) The map F in the Lemma extends to a map from $(\mathbb{C}^n \backslash B_\lambda)/\sim \to \mathrm{Bl}_0\,\mathbb{C}^n$. Here $\partial B_\lambda / S^1$ is mapped to the zero section/exceptional divisor. This map is a symplectomorphism on the dense open set $\mathbb{C}^n \backslash \overline{B}_\lambda$ and on the complement B_λ / S^1. We do not discuss symplectomorphism on all of the VIEW 2 space, because we have not talked about a smooth structure on it.

3. VIEW 3

Here, we adopt a *changed sign convention.* We define moment map by the equation $i_{\xi_M}\omega = -d\Phi_\xi$. This will flip the sign of moment maps.

Consider $\mathbb{C}^{n+1} = \{(w, z) \in \mathbb{C} \times \mathbb{C}^n\}$, with symplectic form $dw \wedge d\overline{w} \oplus dz \wedge d\overline{z}$. Consider an S^1 action on it, where the action of $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is given by

$$(w, z) \mapsto (e^{-\iota\theta} w, e^{\iota\theta} z).$$

A moment map for this action is

$$\Phi(w, z) = \frac{1}{2}(-|w|^2 + \|z\|^2).$$

Any $\epsilon > 0$ is a regular value of the moment map. We will see that the symplectic quotient $\Phi^{-1}(\frac{\epsilon}{2})/S^1$ is the blow-up $\mathrm{Bl}_0\,\mathbb{C}^n$. This is VIEW 3 of the blow-up space.

Proposition 3.1. *The* VIEW 1 *space* $(\mathrm{Bl}_0\,\mathbb{C}^n, \omega_{\sqrt{\epsilon}})$ *is symplectomorphic to* $\Phi^{-1}(\frac{\epsilon}{2})/S^1$.

Proof. The symplectic quotient is a disjoint union

$$\Phi^{-1}(\frac{\epsilon}{2})/S^1 = \Big(\{(w, z) : \|z\|^2 > \epsilon, |w|^2 = \|z\|^2 - \epsilon\} \bigsqcup \{(0, z) : \|z\|^2 = \epsilon\}\Big)/S^1.$$

Notice that the quotient of the first space $\{(w, z) : \|z\|^2 > \epsilon, |w|^2 = \|z\|^2 - \epsilon\}$ is an open dense subset, and the quotient of the second space $\{(0, z) : \|z\|^2 = \epsilon\}$ is a codimension two symplectic submanifold of $\Phi^{-1}(\frac{\epsilon}{2})/S^1$. The quotient of the first space will turn out to be $(\mathrm{Bl}_0\,\mathbb{C}^n \backslash \pi^{-1}(0), \omega_{\sqrt{\epsilon}})$. The quotient of the second space will turn out to be the exceptional divisor in $\mathrm{Bl}_0\,\mathbb{C}^n$.

Consider the map

$$\begin{aligned} f : (\mathbb{C}^n \backslash \{0\}, \omega_{\sqrt{\epsilon}}) &\to \Phi^{-1}(\frac{\epsilon}{2}) \\ z &\mapsto (W, Z) := (\|z\|, \sqrt{\|z\|^2 + \epsilon}\frac{z}{\|z\|}). \end{aligned} \tag{3}$$

The image of f is $\{\mathrm{Re}(w) > 0, \mathrm{Im}(w) = 0\}$, which is a slice of the S^1 action. Notice that the second component of the map is same as F of Lemma 2.1. Using the calculation in that Lemma, we get

$$f^*(dW \wedge \mathrm{d}\overline{W} + dZ \wedge d\overline{Z}) = d(\|z\|) \wedge d(\overline{\|z\|}) + \omega_{\sqrt{\epsilon}} = \omega_{\sqrt{\epsilon}}.$$

After right composing f with the projection $\Phi^{-1}(\frac{\epsilon}{2}) \to \Phi^{-1}(\frac{\epsilon}{2})/S^1$, we get a map $\overline{f} : (\mathbb{C}^n \backslash \{0\}, \omega_{\sqrt{\epsilon}}) \to \Phi^{-1}(\frac{\epsilon}{2})/S^1$. The map $\overline{f}$ is a symplectomorphism onto its image. The image is the quotient of the first set in (3), which is open dense in $\Phi^{-1}(\frac{\epsilon}{2})/S^1$. The map $\overline{f}$ extends continuously to a map

$$\overline{f} : (\mathrm{Bl}_0\, \mathbb{C}^n, \omega_{\sqrt{\epsilon}}) \to \Phi^{-1}(\frac{\epsilon}{2})/S^1.$$

$\overline{f}$ maps the exceptional divisor symplectomorphically to $\{(0, z) : \|z\|^2 = \epsilon\}/S^1$. Check that the derivative of $\overline{f}$ is well-defined on the normal bundle of the exceptional divisor in $\mathrm{Bl}_0\, \mathbb{C}^n$, and so $\overline{f}$ is a differentiable map. Since the pullback of the Kähler form by $\overline{f}$ is equal to $\omega_{\sqrt{\epsilon}}$ on a dense open set, it must agree everywhere. □

Remark 3.2. In View 3, we saw that the blow-up is the symplectic quotient $\Phi^{-1}(\frac{\epsilon}{2})$, for some $\epsilon > 0$. What is the quotient on negative levels? We claim that for any $\epsilon > 0$, the quotient $\Phi^{-1}(\frac{\epsilon}{2})/S^1$ is symplectomorphic to $\mathbb{C}^n$. This is seen as follows. On the set

$$\Phi^{-1}(\frac{\epsilon}{2}) = \{(z, w) : -|w|^2 + \|z\|^2 = -\epsilon\},$$

$|w|$ is positive. Therefore, the set $\{(w, z) : \mathrm{Re}(w) > 0, \mathrm{Im}(w) = 0\} \cap \Phi^{-1}(-\frac{\epsilon}{2})$ is a global slice of the S^1 action on $\Phi^{-1}(-\frac{\epsilon}{2})$. This slice is the image of the map

$$\begin{aligned} f : (\mathbb{C}^n, \omega_{std}) &\to \Phi^{-1}(-\frac{\epsilon}{2}) \\ z &\mapsto (W, Z) := (\sqrt{\|z\|^2 + \epsilon}, z). \end{aligned}$$

Further, this map satisfies $f^*(dW \wedge d\overline{W} + dZ \wedge d\overline{Z}) = dz \wedge d\overline{z}$. Therefore, there is a symplectomorphism $\Phi^{-1}(\frac{\epsilon}{2})/S^1 \simeq (\mathbb{C}^n, \omega_{std})$.

This discussion shows that crossing 0, which is a critical value of Φ, has the effect of changing the symplectic quotient by a blow-up. In general, crossing critical levels changes the symplectic quotient by birational transformations. We do not get into details here.