Symmetry and Duality

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What are symmetries and why are they important?

Most useful concept in Physics ….

Best theoretical models of natural — **Standard Model** & **GTR** — are based on symmetries seen in nature.

Continuous (tuneable parameters) : Poincare, Conformal; Gauge, GCT, …

Continuous: Local symmetries vs Global symmetries

Discrete (not tuneable): CPT, Lattice, …

More symmetry —> More control

Typically physical systems admit more than one description.

Such equivalences >—< **Dualities**
— could be non-trivial, counter intuitive and useful

Toolbox: Field Theory, Groups and Representations ..

In this talk:

Symmetries of free Newtonian particle — Galilean group

Symmetry of free relativistic particle — Poincare group

Symmetries of Maxwell’s EM theory — Conformal group

Hint about holography …

* Abc

Part I
Newtonian Mechanics

Classical Mechanics is the theory to describe dynamics of slowly moving particles, objects etc acted on by forces.

Consider a particle of mass m experiencing a force $\vec{F}$

$$\vec{F} = -\vec{\nabla} V(\vec{x})$$

Conservative Force.

The system is better described by the Lagrangian functional

$$L(\vec{x}(t), \dot{\vec{x}}(t), t)$$

Then consider the action for a path between $\vec{x}(t_i)$ and $\vec{x}(t_f)$

$$S = \int_{t_i}^{t_f} dt \; L(\vec{x}(t), \dot{\vec{x}}(t), t)$$

Hamilton's Principle: $\delta\, S = 0$

under variations of path with $\delta\vec{x}(t_i) = \delta\vec{x}(t_f) = 0$

Leads to the famous Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}_i}\right) - \frac{\delta L}{\delta x_i} = 0$$

For a Newtonian particle in a potential

$$L = \frac{1}{2} m \, \dot{x}^2 - V(\vec{x})$$

Lagrangian is not unique — $L_1 \text{ and } L_2$

give rise to same EOM if

NB

$$L_1 - L_2 = \frac{d}{dt} f(\vec{x}(t), t)$$

Simple examples: Free particle, Harmonic oscillator, charged particle in Electric and Magnetic fields, etc.

In physics symmetries are: Transformations $x_i(t) \to x_i'(t)$

that give rise to form-invariant EOM.

The free particle of course has the maximum symmetry.

To describe its motion using Cartesian coordinates

$$x_i(t) \quad i = 1, 2, 3.$$

But what is $\vec{x}(t)$?

It is the position vector of the particle
in your frame of reference at time t in your watch.

The equation of motion is $\vec{F} := m\,\ddot{\vec{x}}(t) = 0$

The equation has many symmetries:

Under $$x_i(t) \to x'_i(t)$$

the equation has to be form-invariant.

Strictly speaking the symmetries of the EOM are more than the physical system.

Translations: $$x'_i(t) = x_i(t) + a$$

Boosts: $$x'_i(t) = x_i(t) - v_i\, t$$

Rotations: $$x'_i(t) = R_{ij}\, x_j(t)$$ With $$R^{\mathrm{T}}\, R = 1$$

$$x'_i(t) = R_{ij}\, x_j(t) = x_i(t) + \omega_{ij} x_j + \mathcal{O}(\omega^2)$$

Number of continuous parameters is 9.

Not difficult to check the invariance of the action.

Just one other continuous symmetry: Time translations.

$$t \to t + \tau$$

For this we have $$x_i'(t) = x_i(t-\tau)$$

$$= x_i(t) - \tau\, \dot{x}_i(t) + \mathcal{O}(\tau^2)$$

Composing these symmetries also lead to symmetries.

Set of all symmetry transformations form a group.

For free particle this 10 parameter group is Galilian group.

Symmetries lead to conserved quantities.

Noether theorem ⟶ Conserved currents

Space translations —> Linear Momentum $\vec{P} = \frac{\delta L}{\delta \dot{\vec{x}}}$

Rotations —> Angular Momentum $\vec{L} = \vec{x} \times \vec{P}$

Boosts —> $m\,\vec{x} - \vec{P}\,t$

Time translations —> Total Energy $E = \dot{\vec{x}} \cdot \vec{P} - L$

Ex: Verify conservation using free particle EOM $\ddot{\vec{x}} = 0$

Part II
Relativistic Mechanics

Let us turn to relativistic particle mechanics.

Based on two postulates.

(1) Physical equations are form-invariant in any inertial frame.

(2) Velocity of light in vacuum is same in all inertial frames.

Lagrangian of relativistic free-particle

$$L = -m_0\, c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

c is the velocity of light in vacuum $\quad c \approx 3 \times 10^5 \ \ \mathrm{km/sec}$

To see symmetries of free relativistic particle

$$L\,dt = -m_0\,c\sqrt{c^2\,dt^2 - d\vec{x}\cdot d\vec{x}}$$

$$= -m_0\,c\,\sqrt{-\eta_{\mu\nu}\,dx^\mu\,dx^\nu}$$

$\mu, \nu = 0, 1, 2, 3.$ $\quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ $\quad \boldsymbol{x^0 = c\,t}$

$$ds^2 = \eta_{\mu\nu}\,dx^\mu\,dx^\nu$$ —> Line element.

If we take $\quad x^\mu \to \Lambda^\mu{}_\nu\,x^\nu + a^\mu \quad$ such that

$$\Lambda^\alpha{}_\mu\,\Lambda^\beta{}_\nu\,\eta_{\alpha\beta} = \eta_{\mu\nu}$$

then the action is invariant.

Conserved quantities:

$$\vec{P} = \frac{m_0 \, \dot{\vec{x}}}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}} \qquad\qquad E = \frac{m_0 \, c^2}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}}$$

mass-shell

These satisfy:

$$-\frac{E^2}{c^2} + \vec{P} \cdot \vec{P} = m_0^2 \, c^2$$

$$p_\mu = (E/c, \, \vec{p}) \qquad \text{—>} \qquad p_\mu p^\mu - m_0^2 \, c^2 = 0$$

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu \quad \& \quad p_\mu \to p_\nu \Lambda^\nu{}_\mu$$

Successive application —> a group.

$$(\Lambda_2, a_2) \cdot (\Lambda_1, a_1) = (\Lambda_2 \, \Lambda_1, \Lambda_2 \, a_1 + a_2)$$

The Poincare Group

Simplest representation: $\begin{pmatrix} \Lambda_{4\times4} & a_{4\times1} \\ 0_{1\times4} & 1 \end{pmatrix}$

Lorentz group: $\{\Lambda \in GL(4,\mathbb{R}) : \quad \Lambda^{\mathrm{T}} \eta \Lambda = \eta\}$

Component connected to identity? *SO(1,3)* .

Mostly we work with infinitesimal transformations …

$$\Lambda \approx 1 + M$$

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta \implies M^{\mathrm{T}}\eta + \eta\, M = 0$$

The space of *M* can be spanned by six matrices

$$M_{\alpha\beta} = -M_{\beta\alpha}$$

$$(M_{\alpha\beta})^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\,\eta_{\nu\beta} - \delta^{\mu}_{\beta}\,\eta_{\nu\alpha}$$

This is a solution to the Lie algebra of *SO(1,3).*

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \eta_{\alpha\delta}L_{\beta\gamma} - \eta_{\alpha\gamma}L_{\beta\delta} + (\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta)$$

L_{ij} generate rotations

L_{0i} generate Lorentz boosts.

Lie algebra of Poincare includes P_α with

$$[L_{\mu\nu}, P_\sigma] = \eta_{\nu\sigma}\, P_\mu - \eta_{\mu\sigma}\, P_\nu$$

$$= (M_{\mu\nu})^\lambda{}_\sigma\, P_\lambda$$

Another useful way to think about Poincare symmetry:
Symmetry of line element $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$

Under $x^\mu \to x^\mu + \xi^\mu(x)$

$$ds^2 \to ds^2 + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)\, dx^\mu\, dx^\nu + \cdots$$

Setting $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$

—> $\xi^\mu = a^\mu + \omega^\mu{}_\nu\, x^\nu, \quad \text{with} \;\; \omega_{\mu\nu} + \omega_{\nu\mu} = 0$

The vector fields: $\xi^\mu \, \partial_\mu$ —> Killing vectors.

These provide another representation of interest — diff. operators

$$L_{\mu\nu} \to x_\mu \, \partial_\nu - x_\nu \, \partial_\mu \qquad\qquad P_\mu \to \partial_\mu$$

$$\xi^\mu \, \partial_\mu = a^\mu \, P_\mu + \frac{1}{2}\omega^{\mu\nu} \, L_{\mu\nu}$$

Representations of Lorentz and Poincare algebra play huge role in field theories.

Part - III
Symmetries of Electrodynamics

Charged particles, currents and magnets interact through Electric and Magnetic fields.

$\vec{E}(\vec{x},t)$ $\vec{B}(\vec{x},t)$

One works with Electric and Magnetic potentials:

$\phi(\vec{x},t)$ $\vec{A}(\vec{x},t)$

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

The EOM of $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$

Maxwell's Equations

In empty space

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \qquad \boxed{\vec{\nabla} \cdot \vec{E} = 0}$$

$$\boxed{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0} \qquad \boxed{\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = 0}$$

Written in terms of $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$

four equations are satisfied identically.

Simplest solutions are light waves moving at velocity *c* !!

—> Famous examples of Lorentz covariant equations.

This theory has much bigger symmetry
—> Gauge and Conformal symmetries .

Gauge symmetry

The dynamical variables are $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$

Not unique … Because of gauge symmetry:

$$\phi \to \phi - \partial_t \chi, \quad \vec{A} \to \vec{A} + \vec{\nabla}\chi$$

Moving over to Lorentz covariant notation — define:

$$A_\mu = (-\frac{1}{c}\,\phi,\ \vec{A}) \qquad \partial_\mu = (\frac{1}{c}\partial_t, \vec{\nabla})$$

More natural to think of $\mathbf{A} := A_\mu\, dx^\mu = -\phi\, dt + A_i\, dx^i$

In terms of this the gauge symmetry is: $\mathbf{A} \to \mathbf{A} + d\chi$

Then its exterior derivative $\mathbf{F} = d\mathbf{A} := \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

Finally $\partial_\mu F^{\mu\nu} = 0 \leftrightarrow d * \mathbf{F} = 0$ Maxwell's equations.

Under Poincare transformations

$$\delta A_\mu(x) = \tfrac{1}{2}\omega^{\alpha\beta} \left[(M_{\alpha\beta})^\nu{}_\mu \, A_\nu(x) - (x_\alpha \partial_\beta - x_\beta \, \partial_\alpha) A_\mu(x) \right]$$

$$-a^\alpha \, \partial_\alpha \, A_\mu(x)$$

Electric-Magnetic duality ….

$$\frac{\vec{E}}{c} \to \vec{B}, \quad \vec{B} \to -\frac{\vec{E}}{c}$$

A perfect symmetry of Maxwell’s equations.

$$\mathbf{F} \leftrightarrow *\mathbf{F}$$

Lifts to a very important Quantum duality
of a more symmetric theory …

Construction of a Lagrangian for electrodynamics …

* Has to respect gauge symmetry and Poincare symmetry.

A unique candidate (upto boundary terms) !!!

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad\qquad \mathcal{S} = -\frac{1}{4} \int d^4x \; F_{\mu\nu} F^{\mu\nu}$$

Euler-Lagrange $$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0$$

—> $$\partial_\mu F^{\mu\nu} = 0$$

Part - IV
Conformal group

Turns out this action has conformal symmetry.

Coordinate transformations that leave the Poincare invariant metric $\eta_{\mu\nu}$

Invariant upto a conformal factor.

Consider an infinitesimal coordinate trans.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x)$$

Under this a metric transforms as: $g_{\mu\nu}(x) \to g'_{\mu\nu}(x)$

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - (g_{\mu\alpha}\,\partial_\nu \xi^\alpha + g_{\nu\alpha}\,\partial_\mu \xi^\alpha + \xi^\alpha\,\partial_\alpha g_{\mu\nu}) + \cdots$$

For a conformal transformation we demand

$$g'_{\mu\nu}(x) = e^{-\frac{2}{d}\sigma(x)} g_{\mu\nu}(x)$$

Working with Cartesian coordinates for Minkowski space:

CKV eqn.

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} \partial_\alpha \xi^\alpha \ \eta_{\mu\nu}$$

* for *d=1* and *d=2* CKV Eqn. admits infinitely many solutions.

In any *d>2* there are finitely many solutions.

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu \, x^\nu + \lambda \, x^\mu + x^2 I^\mu_\nu(x) \, b^\nu$$

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \qquad\qquad I^\mu_\nu(x) = \delta^\mu_\nu - \frac{2}{x^2} x^\mu \, x_\nu$$

$$\partial_\alpha \xi^\alpha = d \, (\lambda - 2 \, b_\alpha \, x^\alpha)$$

Corresponding algebra of vector fields $\quad \xi^\mu \, \partial_\mu$

can be computed using 6-dimensional matrices.

$$W^A{}_B = \begin{pmatrix} \omega^\mu{}_\nu & a^\mu - b^\mu & a^\mu + b^\mu \\ -a_\nu + b_\nu & 0 & -\lambda \\ a_\nu + b_\nu & -\lambda & 0 \end{pmatrix}$$

$$A, \ B = 0, 1, \cdots, 6. \qquad X^A = (x^\mu, \ \frac{1}{2}(1 - x^2), \frac{1}{2}(1 + x^2))$$

$$W_{AB} = g_{AC} W^C{}_B = -W_{AB}, \quad g_{AB} = \text{diag.}(-1, 1, 1, 1, 1, -1)$$

$$\xi^\mu \, \partial_\mu = \frac{1}{2} W^{AB} (X_A \, \partial_B - X_B \, \partial_A)$$

The algebra is isomorphic to the algebra of *SO(2,4)* group.

The finite coordinate transformations are known.

Lorentz Rotations:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$\frac{dx'^{\mu}}{dx^{\alpha}} = \Lambda^{\mu}{}_{\alpha}$$

Translations:

$$x'^{\mu} = x^{\nu} + a^{\mu}$$

$$\frac{dx'^{\mu}}{dx^{\alpha}} = \delta^{\mu}_{\alpha}$$

Dilatation:

$$x'^{\mu} = e^{\lambda}\, x^{\mu}$$

$$\frac{dx'^{\mu}}{dx^{\alpha}} = e^{\lambda}\, \delta^{\mu}_{\alpha}$$

Special Conformal Transformations: $\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} + b^{\mu}$

$$x'^{\mu} = \frac{x^{\mu} + b^{\mu}\, x^{2}}{1 + 2\, b \cdot x + b^{2}\, x^{2}}$$

$$\frac{dx'^{\mu}}{dx^{\alpha}} = \frac{1}{1 + 2\, b \cdot x + b^{2} x^{2}} I^{\mu}{}_{\sigma}(x') I^{\sigma}{}_{\nu}(x)$$

NB: Upto conformal factors Jacobians are Lorentz rotations!!

One obtains the following transformation laws:

$$F'_{\mu\nu}(x') = \frac{dx^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu}F_{\alpha\beta}(x)$$

Counting the powers of conformal factors
—> Conformal Invariance of Maxwell's action.

Can be used to solve problems in classical Electrodynamics.

Admits a conserved symmetric traceless
Energy-Momentum Tensor

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu\beta}\eta^{\alpha\beta} - \frac{1}{4}F^2\,\eta_{\alpha\beta}$$

Supersymmetric *N=4* Yang-Mills theory
is exactly conformal invariant at Quantum level.

A geometry that has the same symmetries is
5d Anti-DeSitter space.

Consider $\mathbb{R}^{2,4}$ with metric $ds^2 = \eta_{AB}\, dX^A\, dX^B$

Consider the time-like hyper-surface

$$-X_5^2 - X_0^2 + X_1^2 + \cdots + X_4^2 = -L^2$$

This is the space-time AdS_5

Holography is the statement of equivalence
between gravity in. AdS_5 and N=4 SCFT.

Summary …

A rich variety of symmetries play important roles in various physical contexts.

Bigger the symmetry algebra more the control over the theory.

Newtonian mechanics is governed by Galilean symmetry.

Special relativistic mechanics by Lorentz and Poincare.

Maxwell’s electrodynamics in free space has even bigger symmetry — Gauge & Conformal Symmetry.

Different symmetry frames give equivalent descriptions of the same physical systems -> Duality.

Further comments …

Breaking patten of global symmetries can be used to study phase transitions, to contain the matter content of QFT, etc.

Break down of local/gauge symmetries often lead to unphysical theories.

Symmetry considerations lead to remarkably dualities between gravitational and non-gravitational theories.

Developing the dictionary of such a duality involves detailed considerations of consequences of symmetries on both sides ..

A topic of active research at IMSc ….

Thank you.