Homework 9. October 25, 2021.

1. Consider the principal $(\mathbb{C}^\times)$-bundle

$$\mathrm{pr} : \mathbb{C}^n \backslash \{0\} \to \mathbb{P}^{n-1}, (z_1, \dots, z_n) \mapsto [z_1 : \cdots : z_n].$$

Show that

$$\mathrm{pr}^* \omega_{FS} = \frac{i}{2} \partial \overline{\partial} \log(\|z\|^2).$$

Hint :

(a) Show that the form in the rhs is invariant under the action of $\mathbb{C}^\times$ on $\mathbb{C}^n \backslash \{0\}$, and it vanishes on the fibers of pr, and therefore it descends to a form ω'_{FS} on the quotient $(\mathbb{C}^n \backslash \{0\})/\mathbb{C}^\times \simeq \mathbb{P}^{n-1}$.

(b) To show that $\omega_{FS} = \omega'_{FS}$, observe that on any affine chart U_i of $\mathbb{P}^n$, ω_{FS} is equal to the restriction of $\frac{i}{2}\partial\overline{\partial}\log(\|z\|^2)$ to $\{z_i = 1\}$, which is a slice of the $\mathbb{C}^\times$-action.

2. Show that the Fubini-Study form on $\mathbb{P}^{n-1}$ is equal to the reduced symplectic form on the quotient

$$\{z \in \mathbb{C}^n : \|z\| = 1\}/S^1,$$

where we assume that $\mathbb{C}^n$ has the standard symplectic form ω_{std}.
Hint : Prove that the forms $\frac{i}{2}\partial\overline{\partial}\log(\|z\|^2) \in \Omega^2(\mathbb{C}^n \backslash \{0\})$ and $\omega_{std} \in \Omega^2(\mathbb{C}^n)$ are equal when restricted to S^{2n-1}. To do this show that the forms are $U(n)$-invariant, and therefore, it is enough to check for equality at a single point.

3. Write down the reduced symplectic form on the quotient $\{z \in \mathbb{C}^n : \|z = 2\|\}/S^1$. Is it same as the Fubini-Study form?

4. Suppose T has a linear symplectic action on $(\mathbb{C}^n, \omega_{std})$. Show that $\mathbb{C}^n$ splits into T-invariant symplectic subspaces $\oplus_i V_i$, and the T-action on V_i is given by

$$v \xrightarrow{t \in T} \lambda_i(t) v,$$

where $\lambda_i \in T \to \{z \in \mathbb{C} : |z| = 1\}$ is a homomorphism and $\dim_{\mathbb{R}}(V_i) = 2$.
Hint: Use Schur's Lemma, see notes below.

5. For a torus $T = (S^1)^n$ the *integer lattice* of the Lie algebra $\mathfrak{t}$ is defined as

$$\mathfrak{t}_{\mathbb{Z}} := \{\xi \in \mathfrak{t} : e^{2\pi\xi} = \mathrm{Id}\}.$$

and

$$\mathfrak{t}_{\mathbb{Z}}^\vee := \{\eta \in \mathfrak{t}^\vee : \eta(\xi) \in \mathbb{Z} \ \forall \xi \in \mathfrak{t}_{\mathbb{Z}}\}.$$

In problem 4, show that the weights of the action $\mu_i := d\lambda_i : \mathfrak{t} \to Lie(S^1)$ correspond to points in $\mathfrak{t}_{\mathbb{Z}}^\vee$. The moment map for the action is

$$\Phi(v) = \frac{1}{2} \sum_i \|v_i\|^2 \mu_i,$$

where v decomposes as $v = v_1 + \cdots + v_n$, $v_i \in V_i$. Show that if the T-action is effective, then $\dim(T) \leq n$. In the particular case of $\dim(T) = n$, show that the set $\{\mu_i\}_i$ is a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}^\vee$.

Some notes for problem 4: Suppose V is a complex vector space with a Hermitian inner product. A unitary representation of a compact Lie group G is given by a homomorphism $\rho : G \to U(V)$. A unitary representation is a direct sum of irreducible representations $V = \oplus_{i=1}^k V_i$ – the proof is same as that of finite group representations, see Corollary 1.6 in the book 'Representation theory' by Fulton-Harris. The subspace V_i being a sub-representation of V means that the group action on V leaves V_i invariant. Irreducibility of V_i means that there is no non-trivial subspace of V_i which is invariant under the action of G.

Theorem 1 (Schur's Lemma, Theorem 1.7 in Fulton-Harris) *Let G be a compact Lie group and suppose V and W are irreducible representations of G, and $\phi : V \to W$ is a G-module homomorphism. Then,*

(a) either $\phi = 0$ or it is an isomorphism.

(b) If $V = W$, then $\phi = \lambda \, \mathrm{Id}_V$, where $\lambda \in \mathbb{C}$.

One can conclude from Schur's Lemma that the decomposition into irreducible representations discussed above is unique.

Next we will show that an irreducible representation V of a compact torus T is one (complex)-dimensional. Any element $t \in T$ acts on V by a T-invariant homomorphism, and hence using Schur's Lemma, it is just scalar multiplication. Therefore, the action of T is given by

$$t.v = \lambda(t)v, \quad v \in V, t \in T,$$

where $\lambda : T \to \mathbb{C}$ is a homomorphism. By compactness of T, we can argue that the image is contained in $\{z : |z| = 1\}$. This shows that $\dim_{\mathbb{C}}(V) = 1$.