

Homework 6. October 2, 2021.

1. In class we showed that $\pi_1(SU(n)) = \{\text{Id}\}$ for all $n \in \mathbb{N}$. Use this fact to conclude that $\pi_1(U(n)) = \mathbb{Z}$.
2. Fill in the details of the following proof from class : Suppose U, V are oriented surfaces such that

$$V = [0, 1] \times (0, 1), \quad U \cup V \simeq U + 1\text{-cell attached.}$$

In other words, $U \cap V$ has two connected components each of which deformation retracts to $\{0\} \times (0, 1) \subset V$ and $\{1\} \times (0, 1) \subset V$. Let $E \rightarrow U \cup V$ be a complex vector bundle. Show that if $E|_U$ is trivial, $E|(U \cup V)$ is also trivial.

3. In class we defined the holomorphic line bundle $\mathcal{O}(k) \rightarrow \mathbb{P}^1$ whose first Chern number is $k \in \mathbb{Z}$. Show that $\mathcal{O}(k)$ has non-zero holomorphic sections if and only if $k \geq 0$.

HINT

- Let $\sigma : \mathbb{P}^1 \rightarrow \mathcal{O}(k)$ be a holomorphic section. On any local trivialization $E|_U \simeq U \times \mathbb{C}$, the section $\sigma|_U : U \rightarrow \mathbb{C}$ is a holomorphic function and has a power series expansion.
- Consider the cover of $\mathbb{P}^1 = U_0 \cup U_1$, $U_0 := \mathbb{C}$, $U_1 := \mathbb{P}^1 \setminus \{\infty\}$. Examine what happens to the power series $\sigma|_{U_0}$ in the trivialization $U_1 \times \mathbb{C}$. The latter Laurent series should have a removable singularity at ∞ in order for σ to be a globally defined holomorphic section.