Homework 5. September 20, 2021.

1. Show that

$$GL(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) = O(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) = U(n).$$

   Hint : If you get stuck, refer to Lemma 2.19 in Introduction to Symplectic topology by McDuff-Salamon.

2. Suppose a Lie Group G acts transitively on a manifold M, and suppose the isotropy group at m is $G_m \subset G$. Show that the map

$$f : G/G_m \to M, \quad gG_m \mapsto gm$$

   is a diffeomorphism. You may assume the following : $G_m \subset G$ is a closed subgroup of G, and G/G_m is a smooth manifold.
   Hint: Show that f is bijective and has constant rank. Conclude that it is a diffeomorphism by Proposition 6.5 in Introduction to Manifolds by John M. Lee.
   Remark : A manifold with a transitive action of a Lie group is called a *homogeneous space.*

3. Let $\mathcal{J}(\mathbb{R}^{2n}, \omega_{std})$ be the set of (linear) complex structures on $\mathbb{R}^{2n}$ that are compatible with ω_{std}. Show that $\mathrm{Sp}(\mathbb{R}^{2n})$ has a transitive action on $\mathcal{J}(\mathbb{R}^{2n}, \omega_{std})$, and the isotropy group at J_{std} is $U(n)$.