Homework 4. September 9, 2021.

1. Let $E \to M$ be a vector bundle with fiber-wise symplectic form ω. Then $(E, -\omega)$ is also a symplectic vector bundle. Is there an isomorphism $(E, \omega) \to (E, -\omega)$ of symplectic vector bundles?

2. Suppose $E_1 \to \mathbb{P}^1$, $E_2 \to \mathbb{P}^1$ are symplectic vector bundles given by transition functions

$$\Phi^1_{01}, \Phi^2_{01} : U_0 \cap U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$$

respectively. Here $U_0 := B_R \subset \mathbb{P}^1$, $U_1 := \mathbb{P}^1 \backslash B_{1/R}$ for some $R > 1$. Show that if E_1 and E_2 are isomorphic vector bundles then Φ^1_{01}, Φ^2_{01} induce the same homomorphism on fundamental groups, i.e.

$$[\Phi^1_{01}] = [\Phi^2_{01}] : \pi_1(U_0 \cap U_1) \to \pi_1(\mathrm{Sp}(\mathbb{R}^{2n})).$$

3. (Co-isotropic neighborhood theorem) Let X be a co-isotropic submanifold in symplectic manifolds (M_1, ω_1), (M_2, ω_2) with inclusion maps

$$i_1 : X \to (M_1, \omega_1), \quad i_2 : X \to (M_2, \omega_2).$$

Further suppose that $i_1^* \omega_1 = i_2^* \omega_2$. Then show that a neighborhood of X in M_1 is symplectomorphic to a neighborhood of X in M_2. You may assume that both M_1, M_2 are equipped with compatible almost complex structures J_1, J_2.

Hint 1 The proof is similar to the proof of the Lagrangian neighborhood theorem.

Hint 2 The normal bundle $N_{M_k} X$ is isomorphic to $J_k(\ker(\omega_k | TX))$ for $k = 1, 2$. Find an isomorphism ϕ between $J_1(\ker(\omega_1 | TX))$ and $J_2(\ker(\omega_2 | TX))$ such that $\mathrm{Id}_{TX} \oplus \phi : TM_1 | X \to TM_2 | X$ is an isomorphism of symplectic vector bundles.

Remark For $k = 1, 2$, there is a decomposition

$$TX = (TX \cap J_k(TX)) \oplus \ker(\omega_k | TX).$$

The form ω_k is symplectic on the first summand $(TX \cap J_k(TX))$, and it is zero on the second summand.