Homework 3. September 2, 2021.

1. We have seen the following result earlier :

   Theorem 1 *Let (V, ω) be a vector space with a linear symplectic form ω. Then V is even-dimensional and there exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that*

   $$\omega = \sum_i e_i^* \wedge f_i^*$$

   We now prove an analogous result for a family of symplectic vector spaces.

   Let $U \subset \mathbb{R}^m$ be a contractible open set and $E \to U$ be a vector bundle of rank $2n$. (Necessarily E is a trivial bundle and therefore there is a bundle isomorphism $E \to U \times \mathbb{R}^{2n}$.) Let ω be a fiber symplectic form on E. That is, for any $x \in U$

   $$\omega_x : E_x \times E_x \to \mathbb{R}$$

   is a linear symplectic form that varies smoothly with x. Show that there are sections $e_i : U \to E$, $f_i : U \to E$ such that for any point $x \in U$, $\{e_1(x), \ldots, e_n(x), f_1(x), \ldots, f_n(x)\}$ is a basis, and the symplectic form is

   $$\omega = \sum_i e_i^* \wedge f_i^*.$$

   Here $\{e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*\}$ is the dual basis of $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ point-wise. So, note that e_i^*, f_i^* are sections of the dual bundle $E^* \to X$.

   (Help : If you are unfamiliar with vector bundles, refer to p66-68 in Principles of Algebraic Geometry by Griffiths and Harris for definitions and a summary of results that you can use here. The book describes complex vector bundles, but analogous results hold for real vector bundles.)

2. Let (M, ω) be a symplectic manifold and $X \subset M$ be a Lagrangian submanifold. Then show that there is a canonical bundle isomorphism $NX \to T^*X$. (The word 'canonical' means you do not have to make any choices to construct the map.) Hint : Refer to section 9.1 in Cannas da Silva.