Homework 2. August 26, 2021.

1. Let M be a compact manifold, and let $v_t \in \mathrm{Vect}(M)$, $t \in [0,1]$ be a vector field that varies smoothly with time. Prove that the vector field generates a flow $\rho_t : M \to M$, $t \in [0,1]$ that satisfies

$$\frac{d}{dt}\rho_t(m) = v_t(\rho_t(m)).$$

   You may use the following result on the existence of ODEs.

   Theorem 1 *Let $v \in \mathrm{Vect}(M)$ be a vector field on a manifold M. For any point p there is a neighborhood $U \subset M$ and $\epsilon > 0$ such that the flow of v*

$$\rho_t : (-\epsilon, \epsilon) \times U \to M$$

   exists.

   Hint : A time-dependent vector field on M may be viewed as a vector field $\tilde{v} := \frac{\partial}{\partial t} + v_t$ on $[0,1] \times M$. A vector field on $[0,1] \times M$ can be extended to a smooth vector field on $I \times M$ for any open interval $I \subset \mathbb{R}$ that contains $[0,1]$. We may choose the extension to be either compactly supported or time-independent outside a compact interval . (You do no need to prove this fact.)

2. Assume the definition of the Lie derivative of forms given in class by differentiating the pullback of the flow equation. Prove Cartan's formula. (See p36 in Cannas da Silva for hints.)