**Homework 9. October 25, 2021.**

1. This problem is a repeat of Problem 5 from Homework 9. Using notation from Homework 9, show that if the $T$-action on $\mathbb{C}^n$ is effective, then the set of weights $\mu_i \in \mathfrak{t}_{\mathbb{Z}}^{\vee}$ form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}^{\vee}$.
   HINT: Use the Smith normal form. Refer to Wikipedia for the definition.

2. (Equivariant Darboux theorem) Suppose $T = (S^1)^n$ has a symplectic action on the manifold $(M, \omega)$ and $p \in M$ is a fixed point of the action. Show that the $T$-action on the manifold induces a linear symplectic action on the vector space $(T_pM, \omega_p)$. Further, show that there are $T$-invariant neighborhoods $U \subset T_pM$ and $V \subset M$ of 0 and $p$ respectively, and a $T$-equivariant symplectomorphism

   $$\phi : (U, \omega_p) \to (V, \omega).$$

   (You need not repeat steps that are identical to the proof of the Darboux theorem. It is enough to point out how to adapt the proof of the Darboux theorem to this problem. )

3. Suppose $f : M \to \mathbb{R}$ is a smooth function on a compact manifold $M$, and the constants $\epsilon > 0$, $a < b$ are such that there are no critical points of $f$ in $f^{-1}([a-\epsilon, b+\epsilon])$. Show that the manifolds with boundary $\{f \leq a\}$ and $\{f \leq b\}$ are diffeomorphic.
   HINT: Choose a Riemannian metric on $M$ and define $\mathrm{grad}(f) \in \mathrm{Vect}(M)$ to be dual to the one-form $df$. Observe that the time $(b-a)$ flow of the vector field $\frac{\mathrm{grad}(f)}{|\mathrm{grad}(f)|^2}$ takes $f^{-1}(a)$ to $f^{-1}(b)$.