

Introduction to Symplectic Geometry : Lecture 9

September 13, 2021

Recall : Neighborhoods of Lagrangian submanifolds

Theorem (Lagrangian neighborhood theorem)

Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2) . Then a neighborhood of L in M_1 is symplectomorphic to a neighborhood of L in M_2 .

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- Step 2: There is an isomorphism between the complements

$$\phi : J_1(TL) \rightarrow J_2(TL)$$

satisfying

$$\omega_1(v, w_1) = \omega_2(v, \phi(w_1)) \quad \forall l \in L, v \in T_l L, w_1 \in J_1 T_l L.$$

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Consequently $\phi \oplus (\text{Id}_{TL}) : TM_1|L \rightarrow TM_2|L$ is an isomorphism of symplectic vector bundles.

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- Remark : If $L \subset (M, \omega)$ is a Lagrangian submanifold then a neighborhood of L in M is symplectomorphic to a neighborhood of the zero section in T^*L

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Other neighborhood theorems

Theorem (Co-isotropic neighborhood theorem)

Suppose (X, ω_X) is a co-isotropic submanifold in both (M_1, ω_1) and (M_2, ω_2) . Then a neighborhood of X in M_1 symplectomorphic to a neighborhood of X in M_2 .

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Theorem (Isotropic neighborhood theorem)

Suppose X is an isotropic submanifold in both (M_1, ω_1) and (M_2, ω_2) , and there is an isomorphism of symplectic vector bundles $\nu : TX^{\omega_1}/TX \rightarrow TX^{\omega_2}/TX$. Then a neighborhood of X in M_1 symplectomorphic to a neighborhood of X in M_2 .

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- $T_{\text{Id}} \text{Symp}(M, \omega) = \{v \in \text{Vect}(M) : d(i_v \omega) = 0\} \simeq \{\alpha \in \Omega^1(M) : d\alpha = 0\}$.

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- Claim: Let $M = \mathbb{R} \times S^1$ be the cylinder with the standard symplectic form $dt \wedge d\theta$. Let $\phi : M \rightarrow M$ be a Hamiltonian diffeomorphism. Let $C := \{0\} \times S^1$.

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- Contrast with $\text{Symp}_0(M, \omega)$.