

$$s \in (-\frac{1}{R}, R) \quad S'_s$$

$$f: (-\frac{1}{R}, R) \times S^1 \rightarrow \text{Sp}(\mathbb{R}^{2n})$$

Introduction to Symplectic Geometry : Lecture 9

Consider $f|_{S'_s} \quad \exists F: \mathbb{D}_s^2 \setminus \{0\} \rightarrow \text{Sp}(\mathbb{R}^{2n})$

September 13, 2021

$$F|_{S'_s} = f$$

Extend $f|_{S'_s}$



Introduction to Symplectic Geometry : Lecture 9

September 13, 2021

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Recall : Neighborhoods of Lagrangian submanifolds

Theorem (Lagrangian neighborhood theorem)

Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2) . Then a neighborhood of L in M_1 is symplectomorphic to a neighborhood of L in M_2 .

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- Proof, Step 1 : Choose a compatible almost complex structure J_1 on (M_1, ω_1) . Then $J_1(TL) \subset TM_1|_L$ is a Lagrangian complement of the sub-bundle $TL \subset TM_1|_L$.



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- Step 2: There is an isomorphism between the complements

$$\phi : J_1(TL) \rightarrow J_2(TL)$$

satisfying

$$\omega_1(v, w_1) = \omega_2(v, \phi(w_1)) \quad \forall l \in L, v \in T_l L, w_1 \in J_1 T_l L.$$

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Consequently $\phi \oplus (\text{Id}_{TL}) : TM_1|L \rightarrow TM_2|L$ is an isomorphism of symplectic vector bundles.

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- Proof, Step 3 : Use the neighborhood theorem to construct a symplectomorphism from a neighborhood of L in M_1 to a neighborhood of L in M_2 .

$$(\mathbb{R}^{2n}, \omega_0)$$

$$\langle e_1, \dots, e_n \rangle \subseteq \mathbb{R}^{2n} \quad \text{Lagrangian}$$

Recall X : n -dim smooth mfd
 T^*X has a canonical ω .

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$X \simeq \text{Zero section} \subseteq T^*X$
is Lagrangian

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- Remark : If $L \subset (M, \omega)$ is a Lagrangian submanifold then a neighborhood of L in M is symplectomorphic to a neighborhood of the zero section in T^*L

$$M \supset U_0 \xrightarrow{\phi} U_1 \subset T^*L.$$

Lagrangians 'close' to the zero section in

T^*L are the closed 1-forms on M

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$$M \supset U_0 \xrightarrow{\phi} U_1 \subset T^*L. \quad i: L \rightarrow M$$

$$L_1 \equiv i_1: L \rightarrow M$$

- Assume L is compact. If $L_1 \subset M$ is a Lagrangian that is C^1 -close to L then $\phi(L_1)$ is a closed one-form on L .

$dx_1 \wedge dx_2, \dots, dx_{n-1} \wedge dx_n$ are C^0 -close

Other neighborhood theorems

Theorem (Co-isotropic neighborhood theorem)

Suppose (X, ω_X) is a co-isotropic submanifold in both (M_1, ω_1) and (M_2, ω_2) . Then a neighborhood of X in M_1 symplectomorphic to a neighborhood of X in M_2 .

Other neighborhood theorems

Coisotropic $(T_x X)^\omega \subseteq T_x X$

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$i^* \omega = 0$ is isotropic if $i^* \omega = 0$

Theorem (Isotropic neighborhood theorem)

Suppose X is an isotropic submanifold in both (M_1, ω_1) and (M_2, ω_2) , and there is an isomorphism of symplectic vector bundles

$\nu : \underline{TX}^{\omega_1} / TX \rightarrow \underline{TX}^{\omega_2} / TX$. Then a neighborhood of X in M_1 is symplectomorphic to a neighborhood of X in M_2 .

$$TX \subseteq (TX)^\omega$$



Other neighborhood theorems

$$\rightarrow (T_e L)^* \cong J(T_e L)$$

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$$NX \cong TX^{\omega_1}/TX \oplus K \quad \rightarrow (Ker \nu / TX)^* \cong K$$

Back to symplectomorphisms

- For a symplectic manifold (M, ω) , let $\text{Symp}(M, \omega)$ denote the group of symplectomorphisms.

Ex of symplectomorphism

$(\mathbb{R} \times S^1, dt \wedge d\theta)$?

$$(t, \theta) \mapsto (t, \theta + \theta_0) \quad \theta_0 \in S^1$$
$$\leftarrow (t + t_0, \theta) \quad t_0 \in \mathbb{R}$$

Are these in $\text{Symp}_0(M)$? Yes.

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- Recall : The graph of $\phi \in \text{Symp}(M, \omega)$

$$\Gamma_\phi := \{(m, \phi(m)) : m \in M\} \subset (M \times M, \omega \oplus -\omega)$$

is a Lagrangian submanifold.

In particular,

the diagonal $\Delta = \{(m, m) : m \in M\} \subseteq M \times M$
is Lagrangian

$$\Delta = \Gamma_{\text{Id}_M}$$

Back to symplectomorphisms

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- If ϕ is C^1 -close to identity then its graph Γ_ϕ is C^1 -close to the diagonal $\Delta \subset M \times M$.

Back to symplectomorphisms

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- If ϕ is C^1 -close to identity then its graph Γ_ϕ is C^1 -close to the diagonal $\Delta \subset M \times M$.
- Then (assuming M is compact) ϕ corresponds to a closed one-form α_ϕ in M .

Neighbour of Id_M in $\text{Symp}(M) \cong$ Closed 1-forms on M

Back to symplectomorphisms

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- If ϕ is C^1 -close to Id , it is connected to Id_M by a path of symplectomorphisms, proving the result.

- We have also shown that $T_{\text{Id}} \text{Symp}(M, \omega)$ is the space of closed one-forms on M .

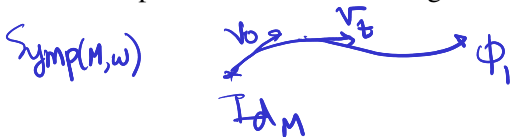
- We have also shown that $T_{\text{Id}} \text{Symp}(M, \omega)$ is the space of closed one-forms on M .
- For a compact manifold M , given a symplectomorphism $\phi_1 \in \text{Symp}_0(M, \omega)$ there is a smooth path

$$\phi_t \in \text{Symp}(M, \omega), t \in [0, 1], \quad \phi_0 = \text{Id}.$$

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- Let $v_t \in \text{Vect}(M)$ be the time-dependent vector field that generates ϕ_t . It satisfies



$\phi_t: M \rightarrow M$ diffeo

Q: What is the condition on v_t so that the flow of v_t is a path of symplectomorphisms?

Ans: $i_{v_t} \omega$ is closed 1-form

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- Let $v_t \in \text{Vect}(M)$ be the time-dependent vector field that generates ϕ_t . It satisfies

$$p_t^* \omega = \omega \Rightarrow \frac{d}{dt} p_t^* \omega = 0 = p_t^* L_{v_t} \omega$$

$$\Leftrightarrow L_{v_t} \omega = 0$$

$$(i_{v_t} d + d i_{v_t}) \omega = 0 \Rightarrow d(i_{v_t} \omega) = 0$$

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- Let $v_t \in \text{Vect}(M)$ be the time-dependent vector field that generates ϕ_t . It satisfies

$$d(i_{v_t} \omega) = 0,$$

and is called a **symplectic vector field**.

$$\forall m \in M$$

$$\omega_m : T_m M \xrightarrow{\cong} T_m^* M$$

$$\omega_m(v_m, \cdot)$$

$$v_m \mapsto \alpha_m$$

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- $T_{\text{Id}} \text{Symp}(M, \omega) = \{v \in \text{Vect}(M) : d(i_v \omega) = 0\} \simeq \{\alpha \in \Omega^1(M) : d\alpha = 0\}$.

canonically

←
?

Show:
Given $\alpha \in \Omega^1(M) \exists v \in \text{Vect}(M) : i_v \omega = \alpha,$

Try non-deg
of ω

Hamiltonian diffeomorphisms

Hamiltonian diffeomorphisms

- If $v \in \text{Vect}(M)$ and $i_{v_*}\omega$ is exact then v_* is called a **Hamiltonian vector field**.

The flow of a Hamiltonian v.f. preserves ω , and $i \circ \cdot$ is a symplectomorphism

Hamiltonian diffeomorphisms

- If $i_{v_t}\omega$ is exact then v_t is called a **Hamiltonian vector field**.
- The flow $\phi_1 : M \rightarrow M$ of a time-dependent Hamiltonian vector field v_t is called a **Hamiltonian diffeomorphism**.

Hamiltonian diffeomorphisms

$$\exists H_t : M \rightarrow \mathbb{R} \quad dH_t = i_{v_t} \omega$$

\Updownarrow

- If $i_{v_t} \omega$ is exact then v_t is called a **Hamiltonian vector field**.
- The flow $\phi_1 : M \rightarrow M$ of a time-dependent Hamiltonian vector field v_t is called a **Hamiltonian diffeomorphism**.
- Thus a Hamiltonian diffeomorphism is given by a time-dependent **Hamiltonian function** $H_t : M \rightarrow \mathbb{R}$, $t \in [0, 1]$, and the generating vector field is

$$i_{v_t} \omega = \underline{d}H_t.$$

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- $H^1(M) = 0$ Closed 1-forms are exact.
- If M is simply connected, $\text{Symp}_0(M, \omega) = \text{Ham}(M, \omega)$,
Flow of v_t Flow of v_t
 $i_{v_t}\omega$ is closed $i_{v_t}\omega$ is exact

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- If M is simply connected, $\text{Symp}_0(M, \omega) = \text{Ham}(M, \omega)$, otherwise they are different.

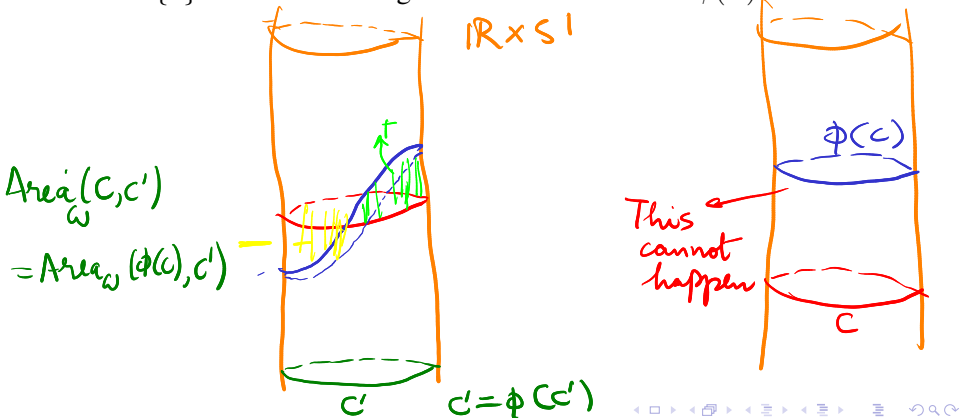
$$\text{Ham}(M, \omega) \neq \text{Symp}_0(M, \omega)$$



- Claim: Let $M = \mathbb{R} \times S^1$ be the cylinder with the standard symplectic form $dt \wedge d\theta$. Let $\phi : M \rightarrow M$ be a Hamiltonian diffeomorphism. Let $C := \{0\} \times S^1$.

$\text{Ham}(M, \omega) \neq \text{Symp}_0(M, \omega)$

- Claim: Let $M = \mathbb{R} \times S^1$ be the cylinder with the standard symplectic form $dt \wedge d\theta$. Let $\phi : M \rightarrow M$ be a Hamiltonian diffeomorphism. Let $C := \{0\} \times S^1$. Then the signed area between C and $\phi(C)$ is zero.



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- Proof: We may assume that the time-dependent Hamiltonian functions H_t have support in a compact set.

ϕ is the time 1 flow of v_t where $dH_t = i_{v_t}\omega$

to get $\bar{H}_t = \eta H_t$.

let $\bar{\phi}_t$ be the flow generated \bar{H}_t

Then $\bar{\phi}_t|_C = \phi_t|_C$

otherwise multiply H_t by a bump function η which is 1 on $\phi_t(C)$

$\text{Ham}(M, \omega) \neq \text{Symp}_0(M, \omega)$

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- Choose a loop $C' = \{t\} \times S^1$ lying outside $\text{supp}(H)$.

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- Choose a loop $C' = \{t\} \times S^1$ lying outside $\text{supp}(H)$. So $\phi(C') = C'$
- A Hamiltonian diffeomorphism preserves ω -area. So, the area between C' and $\phi(C)$ is equal to the area between C' and C .

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- Contrast with $\text{Symp}_0(M, \omega)$.