

Introduction to Symplectic Geometry : Lecture 8

September 8, 2021

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Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a submanifold of a manifold M , and let ω_0, ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|_X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \rightarrow U_1$ which is the identity on X and $\psi^\omega_1 = \omega_0$.*

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- Last time : $X = \{p\}$ gives Darboux theorem.
- $X =$ symplectic submanifold gives the following.

Neighborhood of a symplectic submanifold

Theorem (Symplectic neighborhood theorem)

Let

$$i_1 : (X, \omega_X) \rightarrow (M_1, \omega_1), \quad i_2 : (X, \omega_X) \rightarrow (M_2, \omega_2)$$

be symplectomorphic embeddings. Further suppose there is an isomorphism

$$\nu : N_{M_1}X \simeq N_{M_2}X.$$

of symplectic vector bundles. Then, there are neighborhoods $U_1 \subset M_1$, $U_2 \subset M_2$ of X and a symplectomorphism

$$\psi : (U_1, \omega_1) \rightarrow (U_2, \omega_2), \quad \text{satisfying} \quad \psi|_X = \text{Id}_X.$$

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- Viewing \mathbb{P}^1 as $\mathbb{C} \cup \{\infty\}$, we have a trivialization on charts $U_0 := B_R$, $U_1 := \mathbb{P}^1 \setminus B_{1/R}$ (where $R > 1$)

$$\Phi_0 : E|_{U_0} \simeq U_0 \times \mathbb{R}^{2n}, \quad \Phi_1 : E|_{U_1} \simeq U_1 \times \mathbb{R}^{2n}$$

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- The transition function is a smooth map

$$\Phi_{10} : U_0 \cap U_1 \rightarrow \mathrm{Sp}(\mathbb{R}^{2n}), \quad \Phi_{10}(u) := \Phi_{1,u} \circ \Phi_{0,u}^{-1}.$$

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Theorem

Let $E \rightarrow \mathbb{P}^1$ and $\tilde{E} \rightarrow \mathbb{P}^1$ be symplectic vector bundles given by transition functions

$$\Phi_{10}, \tilde{\Phi}_{10} : U_0 \cap U_1 \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})$$

respectively. The bundles are isomorphic iff there exist maps $\psi_1 : U_1 \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})$ and $\psi_0 : U_0 \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})$ such that

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- The transition function $\Phi_{10} : U_0 \cap U_1 \rightarrow \mathrm{Sp}(\mathbb{R}^{2n})$ induces a homomorphism

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- **Result :** The transition functions $\Phi_{10}, \tilde{\Phi}_{10}$ represent isomorphic bundles iff $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.
- (\Leftarrow) Assume $[\Phi_{10}] = [\tilde{\Phi}_{10}]$. Then there is a map $\psi_1 : U_1 \rightarrow \text{Sp}(\mathbb{R}^{2n})$ that is equal to $\tilde{\Phi}_{10}\Phi_{10}^{-1}$ on $U_0 \cap U_1$.

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Neighborhoods of Lagrangian submanifolds

- Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2) . Is a neighborhood of L in M_1 symplectomorphic to a neighborhood of L in M_2 ?

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- Claim : There is a natural bundle isomorphism $\phi : E_1 \rightarrow E_2$ such that the map $\tilde{\phi} := (\phi + \text{Id}) : TM_1|_L \rightarrow TM_2|_L$ satisfies

$$\tilde{\phi}^* \omega_2 = \omega_1.$$

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- Vector space Claim : Let $(V_1, \omega_1), (V_2, \omega_2)$ be symplectic vector subspaces and suppose $L \subset V_1, L \subset V_2$ is a Lagrangian subspace of both. Further let $U_1 \subset V_1, U_2 \subset V_2$ be Lagrangian subspaces, each of which is complementary to L . Then there is a natural linear isomorphism $\phi : U_1 \rightarrow U_2$ such that $(\phi \oplus \text{Id}_L)^* \omega_2 = \omega_1$.

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- Since we didn't make any choices in constructing ϕ a similar map exists in the bundle (family) case. This proves the Claim.

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Theorem (Lagrangian neighborhood theorem)

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- **Proof :** The proof is a consequence of the Claim above, which shows that there is an isomorphism $\tilde{\phi} : TM_1|L \rightarrow TM_2|L$ of symplectic vector bundles satisfying $\tilde{\phi}|_{TL} = \text{Id}_{TL}$.

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- To finish the proof of the Claim we need to construct the Lagrangian complements of TL in $TM_1|L$ and $TM_2|L$.

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- Additionally, a tame complex structure is **compatible** if

$$\text{(Compatible)} \quad \omega(v, w) = \omega(Jv, Jw) \quad \forall v, w \in V.$$

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- We will show later : on symplectic manifolds tame and compatible almost complex structures exist.

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- Choose compatible almost complex structures J_1, J_2 on TM_1, TM_2 . Take the complements to be $J_1(TL), J_2(TL)$.
- This proves the neighborhood theorem.