Introduction to Symplectic Geometry : Lecture 8

September 8, 2021

Recall

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Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

- Last time : $X = \{p\}$ gives Darboux theorem.

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- Last time : $X = \{p\}$ gives Darboux theorem.
- $X =$ symplectic submanifold gives the following.

Neighborhood of a symplectic submanifold

Theorem (Symplectic neighborhood theorem)

Let

$$i_1 : (X, \omega_X) \to (M_1, \omega_1), \quad i_2 : (X, \omega_X) \to (M_2, \omega_2)$$

be symplectomorphic embeddings. Further suppose there is an isomorphism

$$\nu : N_{M_1}X \simeq N_{M_2}X.$$

of symplectic vector bundles. Then, there are neighborhoods $U_1 \subset M_1$, $U_2 \subset M_2$ *of* X *and a symplectomorphism*

$$\psi : (U_1, \omega_1) \to (U_2, \omega_2), \quad \textit{satisfying} \quad \psi|X = \mathrm{Id}_X \,.$$

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- Viewing $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$, we have a trivialization on charts $U_0 := B_R$, $U_1 := \mathbb{P}^1 \backslash B_{1/R}$ (where $R > 1$)

$$\Phi_0 : E|U_0 \simeq U_0 \times (\mathbb{R}^{2n}, \omega_{std}) \quad \Phi_1 : E|U_1 \simeq U_1 \times (\mathbb{R}^{2n}, \omega_{std})$$

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$$\Phi_0 : E|U_0 \simeq U_0 \times \mathbb{R}^{2n}, \quad \Phi_1 : E|U_1 \simeq U_1 \times \mathbb{R}^{2n}$$

- The transition function is a smooth map

$$\Phi_{10} : U_0 \cap U_1 \to \mathrm{Sp}(\mathbb{R}^{2n}), \quad \Phi_{10}(u) := \Phi_{1,u} \circ \Phi_{0,u}^{-1}.$$

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- The bundle can be reconstructed using the map Φ_{10}.

$$E \simeq \left((U_0 \times \mathbb{R}^{2n}) \amalg (U_1 \times \mathbb{R}^{2n}) \right) / \sim$$

$u \in U_0 \cap U_1$ $\qquad U_0 \times \mathbb{R}^{2n} \ni (u, e) \sim (u, \Phi_{10}(u)e) \in U_1 \times \mathbb{R}^{2n}$

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Last time : Symplectic vector bundles

Theorem

Let $E \to \mathbb{P}^1$ and $\tilde{E} \to \mathbb{P}^1$ be symplectic vector bundles given by transition functions

$$\Phi_{10}, \tilde{\Phi}_{10} : U_0 \cap U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$$

respectively. The bundles are isomorphic iff there exist maps $\psi_1 : U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$ and $\psi_0 : U_0 \to \mathrm{Sp}(\mathbb{R}^{2n})$ such that $\Downarrow$

$$\tilde{\Phi}_{10} = \psi_1 \Phi_{10} \psi_0^{-1}.$$

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- The transition function $\Phi_{10} : U_0 \cap U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$ induces a homomorphism

$$[\Phi_{10}] : \pi_1(U_0 \cap U_1) \to \pi_1(\mathrm{Sp}(\mathbb{R}^{2n}))$$

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- Result : The transition functions $\Phi_{10}, \tilde{\Phi}_{10}$ represent isomorphic bundles iff $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.

$\Phi_{10}, \tilde{\Phi}_{10} : U_0 \cap U_1 \to Sp(2n, \mathbb{R})$

- Result : The transition functions $\Phi_{10}, \tilde{\Phi}_{10}$ represent isomorphic bundles iff $[\Phi_{10}] = [\tilde{\Phi}_{10}]$. ($\Rightarrow$) HW
- ($\Leftarrow$) Assume $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.

Construct Ψ_0, Ψ_1 s.t.

$\tilde{\Phi}_{10} = \Psi_1 \Phi_{10} \Psi_0^{-1}$ (Ψ_0^{-1}: Id)

$\Psi_0 : U_0 \to Sp(2n, \mathbb{R})$

$\Psi_1 : U_1 \to$

Want: $\Psi_1 = \tilde{\Phi}_{10} \Phi_{10}^{-1}$ on $U_0 \cap U_1$

$[\tilde{\Phi}_{10} \cdot \Phi_{10}^{-1}] : \pi_1(U_0 \cap U_1) \to \pi_1(Sp(\mathbb{R}^{2n}))$ is trivial

Last time : Symplectic vector bundles

$f : \partial B^2 \to X$, $\partial B^2 \cong S^1$

$[f] : \pi_1(S^1) \to \pi_1(X)$ is trivial

$\Updownarrow$

$\exists$ continuous $F : B^2 \to X$, $F|_{\partial B^2} = f$

- Result : The transition functions $\Phi_{10}, \tilde{\Phi}_{10}$ represent isomorphic bundles iff $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.
- $(\Leftarrow)$ Assume $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.

$t \in [0,1]$

$h_t : U_0 \cap U_1 \to Sp(2n, \mathbb{R})$

$h_0 = \Phi_{10} \overset{h_t}{\longleftrightarrow} h_1 = \tilde{\Phi}_{10}$

$[\tilde{\Phi}_{10} \Phi_{10}^{-1}] : \pi_1(U_0 \cap U_1) \to \pi_1(Sp(2n, \mathbb{R}))$ is trivial

$\tilde{\Phi}_{10} \Phi_{10}^{-1} \longleftrightarrow \tilde{\Phi}_{10} \tilde{\Phi}_{10}^{-1} = Id$ via $\tilde{\Phi}_{10} h_t$

$\Rightarrow \exists \Psi_1 : U_1 \to Sp(\mathbb{R}^{2n})$ s.t. $\Psi_1 = \tilde{\Phi}_{10}^{-1} \Phi_{10}$ on $U_0 \cap U_1$

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- Result : The transition functions $\Phi_{10}, \tilde{\Phi}_{10}$ represent isomorphic bundles iff $[\Phi_{10}] = [\tilde{\Phi}_{10}]$.
- ($\Leftarrow$) Assume $[\Phi_{10}] = [\tilde{\Phi}_{10}]$. Then there is a map $\psi_1 : U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$ that is equal to $\tilde{\Phi}_{10}\Phi_{10}^{-1}$ on $U_0 \cap U_1$.

Last time : Symplectic vector bundles

$\mathbb{P}^1$ = 0 cell + 2 cell
↓ U_0 U_1

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E is determined by the map

$$[\Phi_{10}] : \pi_1(U_0 \cap U_1) \to \pi_1(Sp(\mathbb{R}^{2n}))$$

Neighborhoods of Lagrangian submanifolds

- Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2). Is a neighborhood of L in M_1 symplectomorphic to a neighborhood of L in M_2?

Recall $L \subseteq (M, \omega)$ is Lagrangian

$\dim L = \frac{1}{2} \dim M$

$\omega|_{TL} \equiv 0$

$\ell \in L$

$TM_1|_L \to TM_2|_L$

$\mathbb{R}^n \subseteq \mathbb{C}^n$

$\{\frac{\partial}{\partial x_i}\}_i$

Complement $\{\frac{\partial}{\partial y_i}\}$

Lagrangian

Neighborhoods of Lagrangian submanifolds

- Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2). Is a neighborhood of L in M_1 symplectomorphic to a neighborhood of L in M_2?

Yes, if we can show that $\exists$ bundle iso $\tilde{\Phi} : TM_1|_L \to TM_2|_L$

s.t. $\tilde{\Phi}^* \omega_2 = \omega_1$

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- Assume the additional information : For $k = 1, 2$, there is a sub-bundle $E_k \subset (TM_k)|L$ that is complementary to TL and which is fiber-wise a Lagrangian subspace.

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- Claim : There is a natural bundle isomorphism $\phi : E_1 \to E_2$

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- Claim : There is a natural bundle isomorphism $\phi : E_1 \to E_2$ such that the map $\tilde{\phi} := (\phi + \text{Id}) : TM_1|L \to TM_2|L$ satisfies

$$\tilde{\phi}^* \omega_2 = \omega_1.$$

$\text{Id} : TL \to TL$

$TM_k|L = E_k \oplus TL$

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- Vector space Claim : Let (V_1, ω_1), (V_2, ω_2) be symplectic vector subspaces and suppose $L \subset V_1$, $L \subset V_2$ is a Lagrangian subspace of both. Further let $U_1 \subset V_1$, $U_2 \subset V_2$ be Lagrangian subspaces, each of which is complementary to L. Then there is a natural linear isomorphism $\phi : U_1 \to U_2$ such that $(\phi \oplus \mathrm{Id}_L)^* \omega_2 = \omega_1$.

Neighborhoods of Lagrangian submanifolds

L, U_1, U_2 Lagrangian

$V_1 = L \oplus U_1 \qquad V_2 = L \oplus U_2$

- Proof of vector space Claim : There are natural maps $U_1 \to L^*$, $U_2 \to L^*$ which are isomorphisms.

$\because$ of non-degeneracy of ω_1, ω_2

$U_1 \to (L \to \mathbb{R})$

$u_1 \mapsto (l \mapsto \omega_1(u_1, l))$

$u_1 \mapsto \omega_1(u_1, \cdot)$

$U_2 \to (L \to \mathbb{R})$

$u_2 \mapsto (l \mapsto \omega_2(u_2, l))$

$\omega_2(u_2, \cdot)$

Neighborhoods of Lagrangian submanifolds

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$$\omega_1(u_1, l) = \omega_2(\phi(u_1), l) \quad \forall l \in L, u_1 \in U_1.$$

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The condition $(\phi \oplus \mathrm{Id}_L)^*\omega_2 = \omega_1$ follows.

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- Since we didn't make any choices in constructing ϕ a similar map exists in the bundle (family) case. This proves the Claim.

Neighborhoods of Lagrangian submanifolds

Theorem (Lagrangian neighborhood theorem)

Suppose L is a Lagrangian submanifold in both (M_1, ω_1) and (M_2, ω_2). Then a neighborhood of L in M_1 symplectomorphic to a neighborhood of L in M_2.

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Theorem (Lagrangian neighborhood theorem)

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- Proof : The proof is a consequence of the Claim above, which shows that there is an isomorphism $\tilde{\phi} : TM_1|L \to TM_2|L$ of symplectic vector bundles satisfying $\tilde{\phi}|TL = \mathrm{Id}_{TL}$.

Neighborhoods of Lagrangian submanifolds

Step1 : Construct Lagrangian complement
Step2 : Prove Claim and obtain $TM|_{L_1} \xrightarrow{\tilde{\Phi}} TM|_{L_2}$
Step3 : Apply neighborhood theorem.

Theorem (Lagrangian neighborhood theorem)

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- Proof : The proof is a consequence of the Claim above, which shows that there is an isomorphism $\tilde{\phi} : TM_1|L \to TM_2|L$ of symplectic vector bundles satisfying $\tilde{\phi}|TL = \mathrm{Id}_{TL}$. → Step 2
- To finish the proof of the Claim we need to construct the Lagrangian complements of TL in $TM_1|L$ and $TM_2|L$. → Step 1

- Apply nbhd thm.

Complex structures on symplectic vector spaces

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- Example : If we view $\mathbb{C}$ as $\mathbb{R}^2$, the standard complex structure on it is $J_{std} = i$.

$\mathbb{C} = (\mathbb{R}^2, J = i)$

$i : \mathbb{R}^2 \to \mathbb{R}^2$

$i^2 = -1$

Complex structures on symplectic vector spaces

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- A **complex structure** on a symplectic vector space (V, ω) is typically required to be **tame**

$$\text{(Tameness)} \quad \omega(v, Jv) > 0 \quad \forall v \in V \backslash \{0\}.$$

[ω_{std} on $\mathbb{R}^{2n}$

J_{std} is ω_{std}-tame

J_{std} on $\mathbb{R}^{2n}$ $\quad \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i} \quad \frac{\partial}{\partial y_i} \mapsto \frac{-\partial}{\partial x_i}$

$\omega_{std}\left(\frac{\partial}{\partial x_i}, J\frac{\partial}{\partial x_i}\right) > 0$]

Complex structures on symplectic vector spaces

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- A **complex structure** on a symplectic vector space (V, ω) is typically required to be **tame**

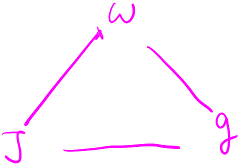
$$\text{(Tameness)} \quad \omega(v, Jv) > 0 \quad \forall v \in V \backslash \{0\}.$$

- Additionally, a tame complex structure is **compatible** if

$$\text{(Compatible)} \quad \omega(v, w) = \omega(Jv, Jw) \quad \forall v, w \in V\,.$$

Complex structures on symplectic vector spaces

- A compatible complex structure gives a positive inner product



Complex structures on symplectic vector spaces

- A compatible complex structure gives a positive inner product $(v, w) \mapsto \omega(v, Jw)$.

Tameness $\Rightarrow$ $(v,v) = \omega(v, Jv) > 0$ if $v \neq 0$

J is compatible $\Rightarrow$ $(v,w) = (w,v)$

Complex structures on symplectic vector spaces

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- A tame complex structure gives a positive inner product

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- On a manifold M a fiberwise complex structure on the tangent space

$$J_x : T_xM \to T_xM, \quad J_x^2 = -\operatorname{Id}$$

is called an **almost complex structure**. on M.

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is called an **almost complex structure**.

- On a symplectic manifold (M, ω) **tame and compatible almost complex structures** are defined in the obvious way.

J on (M, ω) is tame if J_x is tame on $(T_x M, \omega_x)$ $\forall x \in X$

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Complex structures on symplectic vector spaces

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- We will show later : on symplectic manifolds tame and compatible almost complex structures exist.

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Theorem (Lagrangian neighborhood theorem)

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$J_1(TL)$ is a Lagrangian complement of TL

* $J_1(TL)$ is Lagrangian $\because$ $\omega(Jv_1, Jv_2) = \omega(v_1, v_2) = 0$, $v_1, v_2 \in TL$

* Complement.

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- This proves the neighborhood theorem.