# Introduction to Symplectic Geometry : Lecture 7

September 6, 2021

# Last time

**Theorem (Neighborhood theorem in symplectic manifolds)**

*Let $X$ be a ~~compact~~ submanifold of a manifold $M$, and let $\omega_0$ , $\omega_1$ be closed 2-forms on $M$ which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods $U_0$ and $U_1$ of $X$ in $M$ and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on $X$ and $\psi^*\omega_1 = \omega_0$.*

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- Proof: Choose a tubular neighborhood $U \subset M$ of $X$.
- Find a primitive : Let $\omega = \omega_1 - \omega_0$. Since $\omega|(TM|X) = 0$ and $d\omega = 0$, there is a primitive $\mu \in \Omega^1(U)$ such that $d\mu = \omega$ and $\mu_x = 0$ for all $x \in X$.

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- Apply Moser's trick : Find a time-dependent vector field $v_t \in \mathrm{Vect}(U)$, $t \in [0, 1]$ satisfying $-i_{v_t}\omega_t = \mu$. Note $v_t(x) = 0$ for $t \in [0, 1]$, $x \in X$.

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- Note that compactness of $X$ is not required.
- Last time : we proved Darboux's theorem using the neighborhood theorem by setting $X = \textit{point}$.

# Darboux's theorem

**Theorem (Darboux's theorem)**

*Let $(M,\omega)$ be a $2n$-dimensional symplectic manifold. For any point $p$ there is a neighborhood $U$ and coordinates*

$$(x_1,\dots,x_n,y_1,\dots,y_n) : U \to \mathbb{R}^{2n}, \quad p \mapsto 0$$

*such that $\omega|U = \sum_i dx_i \wedge dy_i$.*

# Darboux's theorem

- Proof : Choose a linear symplectomorphism

$$L : T_pM \to T_0\mathbb{R}^{2n} \simeq \mathbb{R}^{2n}.$$

$\phi : U \to \mathbb{R}^{2n}$

$\quad\;\; \Psi$

$\quad\;\; p \mapsto 0$

$d\phi_p : T_pU \to T_0\mathbb{R}^{2n}$

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- There are two symplectic forms on $U_0 \subset M$, namely $\omega_0 := \omega$ and $\omega_1 := \phi^*\omega_{std}$, that satisfy $\omega_0(p) = \omega_1(p)$.

$$U_0 \xrightarrow{\phi} V_0 \subseteq \mathbb{R}^{2n}$$

$$\omega_0 = \omega, \quad \underset{\omega_1}{\underset{\shortparallel}{\phi^*\omega_{std}}} \qquad \omega_{std}$$

$$\omega_0(p) = \omega_1(p)$$

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- By the ~~symplectic~~ neighborhood theorem applied to the submanifold $X = \{p\}$, we conclude that
  - there are neighborhoods $U_1, U_2 \subset U_0$ containing $p$,
  - and a diffeomorphism

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- The chart required by the theorem is $\phi \circ \rho$.

# Neighborhood of a symplectic submanifold

- Question (*): Suppose $(X, \omega_X)$ is a symplecic manifold that is symplectomorphically embedded in $(M_1, \omega_1)$ and $(M_2, \omega_2)$.

$k = 1, 2$

$$i_1 : (X, \omega_X) \to (M_1, \omega_1), \quad i_2 : (X, \omega_X) \to (M_2, \omega_2), \quad \underline{i_k^* \omega_k = \omega_X}.$$

Under what condition is a neighborhood of $i_1(X) \subset M_1$ symplectomorphic to a neighborhood of $i_2(X) \subset M_2$?

Given : $\omega_1|TX = \omega_2|TX$

Additionally we need

$$\omega_1 | \underline{(TM_1)|X} = \omega_2 | \underline{(TM_2)|X}$$

$$\begin{tikzcd}
(TM_1)|X \arrow[rr, "\phi"] \arrow[dr] & & (TM_2)|X \arrow[dl] \\
& X &
\end{tikzcd}$$

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  - $\phi^* \omega_2 = \omega_1$ on the zero section,
  - and $\phi|TX = \mathrm{Id}_{TX}$.

Proof ~~outline~~: There exists a diffeo

$$\begin{tikzcd} M_1 & & M_2 \\ U_1 \arrow[u, phantom, "\cup"] \arrow[rr, "\psi"] & & U_2 \arrow[u, phantom, "\cup"] \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\psi|_X = \mathrm{Id}_X$

s.t. $\forall x \in X$ $\quad d\psi_x = \phi(x)$

$\omega_1, \psi^* \omega_2$ are closed 2 forms on $U_1$ s.t. $\omega_1 = \psi^* \omega_2$ on $TM_1|X$

Apply Nbhd thm.

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- The proof follows from the neighborhood theorem.

$TX^{\omega_1} \simeq N_{M_1} X$

$TX^{\omega_2} \simeq N_{M_2} X$

Enough to supply a bundle map $\phi : TX^{\omega_1} \longrightarrow TX^{\omega_2}$

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- Note that $TM_1|X$ is a 'symplectic vector bundle' on $X$.

# Symplectic vector bundle

- A vector bundle $E \to M$ is a **symplectic vector bundle** if there is a fiber-wise linear symplectic form

$$\omega_x : T_xM \times T_xM \to \mathbb{R}$$

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Eg : $E := M \times (\mathbb{R}^{2n}, \omega_{std})$ is a symp. vector bundle.

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$T_xX \subseteq T_xM$ symp subspace

$T_xX^\omega \subseteq T_xM$ is also a symp subspace

$$TX \to X \qquad TM|X \to X \qquad (TX)^\omega \to X$$

$W \oplus W^\omega = V$

Recall: $W \subseteq (V, \omega)$ symp subspace. $W^\omega$ is also a symp. subspace

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- If $X \subset (M, \omega)$ is a symplectic submanifold, then $TM|X \to X$ is a symplectic vector bundle, and so also is $(TX)^\omega \to X$.
- Remark : There is a natural isomorphism of symplectic vector bundles

$W', W \subseteq (V, \omega)$
$W' \oplus W = V$

$$\underline{NX \to} TX^\omega,$$

$N_x X = T_x M / T_x X$

because there is a fiberwise direct sum $\underline{T_x M} = \underline{T_x X} \oplus \underline{(T_x X)}^\omega$.

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**Theorem (Symplectic neighborhood theorem)**

*Let*

$$i_1 : (X, \omega_X) \to (M_1, \omega_1), \quad i_2 : (X, \omega_X) \to (M_2, \omega_2)$$

*be symplectomorphic embeddings. Further suppose there is an isomorphism*

$$\nu : N_{M_1}X \simeq N_{M_2}X.$$

*of symplectic vector bundles. Then, there are neighborhoods* $U_1 \subset M_1$, $U_2 \subset M_2$ *of* $X$ *and a symplectomorphism*

$$\psi : (U_1, \omega_1) \to (U_2, \omega_2), \quad \textit{satisfying} \quad \psi|X = \mathrm{Id}_X \, .$$

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- Let $U \subset \mathbb{R}^m$ a contracible open set, and let $E \to U$ be a symplectic vector bundle. Then there is a trivialization

$$\Phi : E \simeq U \times (\mathbb{R}^{2n}, \omega_{std}), \quad (u, e) \mapsto (u, \underline{\Phi_u(e)}),$$

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- Any smooth map $\psi : U \to \mathrm{Sp}(\mathbb{R}^{2n})$ will give us a new trivialization

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- Conversely any two trivializations are related by a map $\psi : U \to \mathrm{Sp}(\mathbb{R}^{2n})$.

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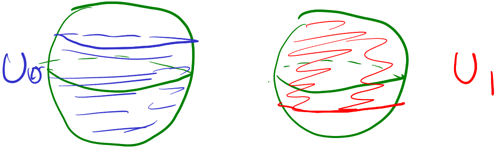
$$\Phi_0 : E|U_0 \simeq U_0 \times (\mathbb{R}^{2n}, \omega_{std}) \quad \Phi_1 : E|U_1 \simeq U_1 \times (\mathbb{R}^{2n}, \omega_{std})$$



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$$\Phi_0 : E|U_0 \simeq U_0 \times \mathbb{R}^{2n}, \quad \Phi_1 : E|U_1 \simeq U_1 \times \mathbb{R}^{2n}$$

- The transition function is a smooth map

$$\Phi_{10} : U_0 \cap U_1 \to \underline{\mathrm{Sp}(\mathbb{R}^{2n})}, \quad \Phi_{10}(u) := \Phi_{1,u} \circ \Phi_{0,u}^{-1}.$$

$$\begin{tikzcd} & E|_{U_0 \cap U_1} \arrow[dl, "\Phi_0"'] \arrow[dr, "\Phi_1"] & \\ U_0 \times (\mathbb{R}^n, \omega_{std}) \arrow[rr, bend left] & & U_1 \times (\mathbb{R}^n, \omega_{std}) \end{tikzcd}$$

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- For a map $\psi_1 : U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$, replacing the trivialization $\Phi_1$ by $\psi_1\Phi_1$ has the effect of changing $\Phi_{10}$ to

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pointwise matrix multiplication

# Symplectic vector bundles

- As an example let's consider a symplectic vector bundle $E \to \mathbb{P}^1$.
- Viewing $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$, we have a trivialization on charts $U_0 := B_R$, $U_1 := \mathbb{P}^1 \backslash B_{1/R}$ (where $R > 1$)

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# Symplectic vector bundles

## Theorem

*Let $E \to \mathbb{P}^1$ and $\tilde{E} \to \mathbb{P}^1$ be symplectic vector bundles given by transtition functions*

$$\Phi_{10}, \tilde{\Phi}_{10} : U_0 \cap U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$$

*respectively. The bundles are isomorphic iff there exist maps $\psi_1 : U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$ and $\psi_0 : U_0 \to \mathrm{Sp}(\mathbb{R}^{2n})$ such that*

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$(\Rightarrow)$ obvious

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$$[\tilde{\phi}_{10}\phi_{10}^{-1}] = 0$$

$$\tilde{\phi}_{10}\phi_{10}^{-1} : U_0 \cap U_1 \longrightarrow Sp(\mathbb{R}^{2n})$$

extends

to $\psi_1 : U_1 \longrightarrow Sp(\mathbb{R}^{2n})$

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- $(\Leftarrow)$ Assume $[\Phi_{10}] = [\tilde{\Phi}_{10}]$. Then there is a map $\psi_1 : U_1 \to \mathrm{Sp}(\mathbb{R}^{2n})$ that is equal to $\tilde{\Phi}_{10}\Phi_{10}^{-1}$ on $U_0 \cap U_1$. Take $\psi_0 = \mathrm{Id}$.