

# Introduction to Symplectic Geometry : Lecture 6

September 3, 2021

# Today's theorem

## Theorem (Neighborhood theorem in symplectic manifolds)

*Let  $X$  be a compact submanifold of a manifold  $M$ , and let  $\omega_0, \omega_1$  be closed 2-forms on  $M$  which are equal and non-degenerate on  $TM|_X$ . Then there exist neighbourhoods  $U_0$  and  $U_1$  of  $X$  in  $M$  and a diffeomorphism  $\psi : U_0 \rightarrow U_1$  which is the identity on  $X$  and  $\psi^*\omega_1 = \omega_0$ .*

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Last time we showed :

## Theorem (Tubular neighborhood theorem)

*Let  $X \subset M$  be a compact submanifold. Then there exists a neighborhood  $U_0 \subset NX$  of the zero section, a neighborhood  $U \subset M$  of  $X$ , and a diffeomorphism  $\psi : U_0 \rightarrow U$  that is equal to  $\text{Id}_X$  on the zero section.*

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In our proof the map  $\psi$  is the exponential map restricted to the normal bundle  $NX$ .

# A primitive in a tubular neighborhood

## Theorem (Construction of primitive)

Let  $U \subset M$  be a tubular neighborhood of a submanifold  $X \subset M$ . Let  $i : X \rightarrow U$  be the inclusion map. Let

$$\omega \in \Omega^2(U), \quad d\omega = 0, \quad i^*\omega = 0.$$

Then there exists

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- By a construction in differential geometry, there exists

$$Q : \Omega^*(U) \rightarrow \Omega^{*-1}(U) \quad \text{such that} \quad dQ + Qd = \text{Id}_U - (i \circ \pi)^*$$

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- For  $\omega \in \Omega^2(M)$  in the theorem,  $d(Q\omega) = \omega$ .

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- $$\begin{aligned} \omega &= \rho_1^* \omega - \rho_0^* \omega = \int_0^1 \left( \frac{d}{dt} \rho_t^* \omega \right) dt \\ &= \int_0^1 \rho_t^* (L_{v_t} \omega) dt \end{aligned}$$

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- Thus  $\mu := Q\omega := \int_0^1 \rho_t^* (i_{v_t} \omega)$

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- $0 = \frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(L_{v_t}\omega_t + \frac{d}{dt}\omega_t)$
- $-L_{v_t}\omega_t = \frac{d}{dt}\omega_t = \omega_1 - \omega_0$  implies  $-d(i_{v_t}\omega_t) = d\mu$ .

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- Finally, set  $U_1 := \rho_1(U_0)$

# Darboux's theorem

## Theorem (Darboux's theorem)

*Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold. For any point  $p$  there is a neighborhood  $U$  and coordinates*

$$(x_1, \dots, x_n, y_1, \dots, y_n) : U \rightarrow \mathbb{R}^{2n}, \quad p \mapsto 0$$

*such that  $\omega|_U = \sum_i dx_i \wedge dy_i$ .*

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- By the symplectic neighborhood theorem applied to the submanifold  $X = \{p\}$ , we conclude that
  - ▶ there are neighborhoods  $U_1, U_2 \subset U_0$  containing  $p$ ,
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- The chart required by the theorem is  $\phi \circ \rho$ .