Introduction to Symplectic Geometry : Lecture 6

September 3, 2021

Today's theorem

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

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Last time we showed :

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists a neighborhood $U_0 \subset NX$ of the zero section, a neighborhood $U \subset M$ of X, and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.

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In our proof the map ψ is the exponential map restricted to the normal bundle NX.

A primitive in a tubular neighborhood

Theorem (Construction of primitive)

do not need X to be compact

Let $U \subset M$ be a tubular neighborhood of a submanifold $X \subset M$. Let $i : X \to U$ be the inclusion map. Let

$$\omega \in \Omega^2(U), \quad d\omega = 0, \quad i^*\omega = 0.$$

Then there exists

$$\mu \in \Omega^1(U), \quad d\mu = \omega, \quad \mu_x = 0 \forall x \in X.$$

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$\mu|(TU|X) = 0$

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- Remark : Let $\pi : U \to X$ be the projection map.

$i^*\omega = \omega|_{TX}$

$i^*\omega = 0 \to$

$$? \left[\begin{array}{l} \omega|TX = 0 \\ \omega|(TM)|X = 0 \end{array} \right.$$

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- Remark : Let $\pi : U \to X$ be the projection map. Then $i \circ \pi : U \to U$ is homotopy equivalent to Id_U.
- By a construction in differential geometry, there exists

$$Q : \Omega^*(U) \to \Omega^{*-1}(U) \quad \text{such that} \quad dQ + Qd = \mathrm{Id}_U^* - (i \circ \pi)^*$$

$: \Omega^*(U) \to \Omega^*(U)$

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$\Rightarrow [\omega] = 0$ in $H^2(U)$

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- For $\omega \in \Omega^2(M)$ in the theorem, $d(Q\omega) = \omega$.

$\boxed{(i \circ \pi)^* \omega = 0}$

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$$\rho_t(x, v) := (x, tv) \quad x \in X, v \in N_xX, t \in [0, 1].$$

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- Note that $\rho_1 = \mathrm{Id}_{NX}$ and $\rho_0 = \pi$.
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$\pi^* i^* \omega$

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- $\underline{\omega = \rho_1^*\omega - \rho_0^*\omega} = \int_0^1 (\frac{d}{dt}\rho_t^*\omega)dt$

$\rho_0^*\omega = \pi^*\omega$
$= 0$

$\rho_0^*\omega = 0$

$\pi^* : \Omega^*(X) \to \Omega^*(U)$

$\rho_1^*\omega = \omega$

$\omega|_{TX} = 0$

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- $\omega = \rho_1^*\omega - \rho_0^*\omega = \int_0^1 (\underline{\frac{d}{dt}\rho_t^*\omega})dt$
$= \int_0^1 \rho_t^* \underbrace{(L_{v_t}\omega)}_{}dt$

$\downarrow$ Cartan

$d i_{v_t}\omega + i_{v_t}\underline{d\omega}$

$\int \underline{\rho_t^*} d(i_{v_t}\omega)\, dt$

$= \int d(\rho_t^* i_{v_t}\omega)\, dt$

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- Thus $\mu := Q\omega := \int_0^1 \rho_t^*(\underline{i_{v_t}\omega})$

$x \in X$ $\quad \rho_t(x) = x$

$v_t(x) = 0$

$(i_{v_t}\omega)_x = 0$

$\Rightarrow \mu_x = 0$

Neighborhood theorem in symplectic manifolds

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- Choose a tubular neighborhood $U \subset M$ of X.
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- 3 Let $\omega_t = (1-t)\omega_0 + t\omega_1$. By shrinking U, we can ensure that ω_t is symplectic on U for all t.

MOSER

$$\boxed{\frac{d\omega_t}{dt} = d\mu_t}$$

HERE $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ non deg on $TM|X$

$\omega_t = \omega_0 = \omega_1$ on $TM|X$

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$$\frac{d\omega_t}{dt} = \omega_1 - \omega_0 =: \omega$$

Primitive construction : $d\omega = 0$

$i^*\omega = 0 \quad i : X \hookrightarrow U$

$\therefore \omega|(TM|X) = 0$

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- $0 = \frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(L_{v_t}\omega_t + \frac{d}{dt}\omega_t)$
- $-L_{v_t}\omega_t = \frac{d}{dt}\omega_t = \omega_1 - \omega_0$ implies $\boxed{-d(i_{v_t}\omega_t) = d\mu.}$

Neighborhood theorem in symplectic manifolds

- We will choose v_t such that $-i_{v_t}\omega_t = \mu$.

ω_t is symp on U. So for any $x \in U$

$$\omega_t : T_xU \xrightarrow[\text{iso}]{} \underline{T_x^*U}$$

μ_x

So $\exists\, v_t \in \mathrm{Vect}(U)$:

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$(0,1) \subseteq \mathbb{R}$

$v = \frac{\partial}{\partial x}$ on $(0,1)$

Does flow of v exist ? on $(0,1)$

$\rho_t : x \mapsto x+t$

$1-t \mapsto$ outside $(0,1)$

So ρ_t does not exist for $t \neq 0$

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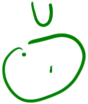
$\mathbb{R}^2$



$v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

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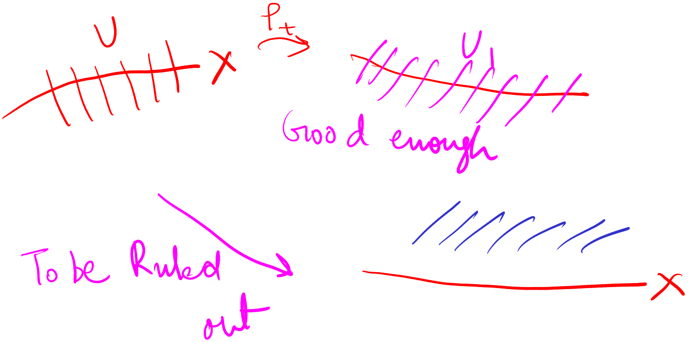

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- For any $x \in X$, $v_t(x) = 0$ for all t. So the flow $\rho_t(x)$ exists for any $x \in X$ and $t \in [0, 1]$. $(\because \mu_x = 0 \ \forall x \in X)$ $\rho_t(x) = x$

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- By the ODE theorem, the set

  $$\mathcal{U} := \{(u, t) \in U \times \mathbb{R} : \rho_t(u) \in U \text{ exists}\}$$

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Neighborhood theorem in symplectic manifolds

$\psi : U_0 \to U_1$

$\psi := \rho_1$

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- Since $X \times [0, 1] \subset U$, there is a neighborhood $U_0 \subset U$ of X such that $U_0 \times [0, 1] \subset \mathcal{U}$.
- Finally, set $U_1 := \rho_1(U_0)$

U_1 is a nbhd of X because $\rho_1|X = Id_X$

Darboux's theorem

Theorem (Darboux's theorem)

Let (M,ω) be a 2n-dimensional symplectic manifold. For any point p there is a neighborhood U and coordinates

$$(x_1,\ldots,x_n,y_1,\ldots,y_n) : U \to \mathbb{R}^{2n}, \quad p \mapsto 0$$

such that $\omega|U = \sum_i dx_i \wedge dy_i$.

$\Updownarrow$

$\exists \ \psi : U \longrightarrow \mathbb{R}^{2n}$ diffeo onto its image

s.t. $\psi(0) = p$ $\qquad \psi^* \omega_{std} = \omega$

Darboux's theorem

- Proof : Choose a linear symplectomorphism

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- There are two symplectic forms on $U_0 \subset M$, namely $\omega_0 := \omega$ and $\omega_1 := \phi^*\omega_{std}$, that satisfy $\omega_0(p) = \omega_1(p)$.

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- By the symplectic neighborhood theorem applied to the submanifold $X = \{p\}$, we conclude that
  - there are neighborhoods $U_1, U_2 \subset U_0$ containing p,
  - and a diffeomorphism

$$\rho : U_1 \to U_2, \quad p \mapsto p$$

  such that $\rho^*(\phi^*\omega_{std}) = \omega$.

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- The chart required by the theorem is $\phi \circ \rho$.