Introduction to Symplectic Geometry : Lecture 5

August 30, 2021

Last time

We proved Moser's theorem.

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0,1]\}$ *is a smooth family of symplectic forms on a compact manifold* M *such that the cohomology class* $[\omega_t] \in H^2(M)$ *is t-independent. Then there exists a family of diffeomorphisms* $\rho_t : M \to M$ *such that* $\rho_t^* \omega_t = \omega_0$.

Last time

We proved Moser's theorem.

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0,1]\}$ is a smooth family of symplectic forms on a compact manifold M such that the cohomology class $[\omega_t] \in H^2(M)$ is t-independent. Then there exists a family of diffeomorphisms $\rho_t : M \to M$ such that $\rho_t^* \omega_t = \omega_0$.

The proof used 'Moser's trick' : Show that a vector field v_t that generates ρ_t can be found by solving the equation $-i_{v_t}\omega_t = \mu$.

Today's theorem

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

Today's theorem

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

We will show that Darboux's theorem is a consequence.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- The normal bundle of a submanifold $X \subset M$ is

$$NX := \cup_{x \in X} N_x X, \quad N_x X = T_x M / T_x X.$$

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- The normal bundle of a submanifold $X \subset M$ is

$$NX := \cup_{x \in X} N_x X, \quad N_x X = T_x M / T_x X.$$

In the presence of a Riemannian metric, it can be identified with the orthogonal complement of TX in $TM|_X$.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- The normal bundle of a submanifold $X \subset M$ is

$$NX := \cup_{x \in X} N_x X, \quad N_x X = T_x M / T_x X.$$

  In the presence of a Riemannian metric, it can be identified with the orthogonal complement of TX in $TM|_X$.
- The proof of the theorem uses the exponential map which we recall.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_mM \to M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_mM \to M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.
- A geodesic satisfies the equation $\nabla_{\gamma'(t)}\gamma'(t) = 0$, which is an ODE in TM.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_m M \to M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.
- A geodesic satisfies the equation $\nabla_{\gamma'(t)}\gamma'(t) = 0$, which is an ODE in TM.
- $\exp_m(v)$ is the time 1 flow of the point $(m, v) \in TM$.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_mM \to M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.
- A geodesic satisfies the equation $\nabla_{\gamma'(t)}\gamma'(t) = 0$, which is an ODE in TM.
- $\exp_m(v)$ is the time 1 flow of the point $(m, v) \in TM$.
- For any $\epsilon \in \mathbb{R}$, the time ϵ flow is $\exp_m(\epsilon v)$.

The exponential map : Existence

- Let $S \subset M$ be a compact submanifold and let

$$\mathbb{D}(TM|S) := \{v \in T_sM : s \in S, |v| \leq 1\}$$

be disk tangent bundle restricted to S.

The exponential map : Existence

- Let $S \subset M$ be a compact submanifold and let

$$\mathbb{D}(TM|S) := \{v \in T_sM : s \in S, |v| \leq 1\}$$

  be disk tangent bundle restricted to S.
- There exists $\epsilon > 0$ such that the time ϵ flow is well-defined for the geodesic flow equation on $\mathbb{D}(TM|S)$.

The exponential map : Existence

- Let $S \subset M$ be a compact submanifold and let

$$\mathbb{D}(TM|S) := \{v \in T_sM : s \in S, |v| \leq 1\}$$

be disk tangent bundle restricted to S.

- There exists $\epsilon > 0$ such that the time ϵ flow is well-defined for the geodesic flow equation on $\mathbb{D}(TM|S)$.
- Therefore $\exp$ is well-defined on $\mathbb{D}_\epsilon(TM|S) := \{v \in TM|S : |v| \leq \epsilon\}$ and

$$\exp : \mathbb{D}_\epsilon(TM|S) \to M$$

is a smooth map.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- For ϵ small enough
$$\exp : \mathbb{D}_\epsilon(NX) \to M$$
is a diffeomorphism onto its image.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- For ϵ small enough
$$\exp : \mathbb{D}_\epsilon(NX) \to M$$
is a diffeomorphism onto its image.
- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- For ϵ small enough
$$\exp : \mathbb{D}_\epsilon(NX) \to M$$
is a diffeomorphism onto its image.
- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity. The theorem follows by the inverse function theorem.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
- *a neighborhood $U \subset M$ of X,*
- *and a diffeomorphism $\psi : U_0 \to U$ that is equal to Id_X on the zero section.*

- For ϵ small enough
$$\exp : \mathbb{D}_\epsilon(NX) \to M$$
is a diffeomorphism onto its image.
- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity. The theorem follows by the inverse function theorem.
- We say that $U \subset X$ is a **tubular neighborhood** of X.