$\psi \in \Omega^2(M)$ exact . there is a unique way of choosing $\mu \in \Omega^1(M)$ $d\mu = \psi$

$d\mu_0 = 0 \quad d(\mu + \mu_0) = d\mu$

Introduction to Symplectic Geometry : Lecture 5

choose μ to be in the 'complement' of closed 1-forms.

August 30, 2021

$[\omega_t] \in H^2(M)$ t-indep

Hodge theory $\Rightarrow \exists\, \mu_t \in \Omega^1(M)$

$\frac{d\omega_t}{dt} = \mu_t$

Last time

We proved Moser's theorem.

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0,1]\}$ *is a smooth family of symplectic forms on a compact manifold* M *such that the cohomology class* $[\omega_t] \in H^2(M)$ *is t-independent. Then there exists a family of diffeomorphisms* $\rho_t : M \to M$ *such that* $\rho_t^* \omega_t = \omega_0$.

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The proof used 'Moser's trick' : Show that a vector field v_t that generates ρ_t can be found by solving the equation $-i_{v_t}\omega_t = \mu$.

Today's theorem

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods U_0 and U_1 of X in M and a diffeomorphism $\psi : U_0 \to U_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

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We will show that Darboux's theorem is a consequence.

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- *a neighborhood $U_0 \subset NX$ of the zero section,*
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$$NX := \cup_{x \in X} N_x X, \quad N_x X = T_x M / T_x X.$$

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- The proof of the theorem uses the exponential map which we recall.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_m M \to M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.

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- A geodesic satisfies the equation $\nabla_{\gamma'(t)}\gamma'(t) = 0$, which is an ODE in TM.
- $\exp_m(v)$ is the time 1 flow of the point $(m, v) \in TM$.
- For any $\epsilon \in \mathbb{R}$, the time ϵ flow is $\exp_m(\epsilon v)$.

The exponential map : Existence

- Let $S \subset M$ be a compact submanifold and let

$$\mathbb{D}(TM|S) := \{v \in T_sM : s \in S, |v| \leq 1\}$$

be disk tangent bundle restricted to S.

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- There exists $\epsilon > 0$ such that the time ϵ flow is well-defined for the geodesic flow equation on $\mathbb{D}(TM|S)$.
- Therefore $\exp$ is well-defined on $\mathbb{D}_\epsilon(TM|S) := \{v \in TM|S : |v| \leq \epsilon\}$ and
$$\exp : \mathbb{D}_\epsilon(TM|S) \to M$$
is a smooth map.

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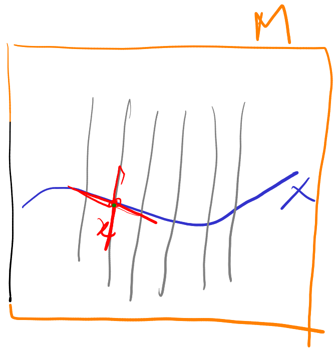
- For ϵ small enough
$$\exp : \mathbb{D}_\epsilon(NX) \to M$$
is a diffeomorphism onto its image.

Given $\exp : \mathbb{D}_\varepsilon(NX) \to M \quad \to (1)$

is well-defined

For small enough $\varepsilon > 0$, (1) is a diffeo onto its image.



Proof : Claim $\forall x \in X \subset NX$ (zero section)

$\exp_x(0) = x \qquad d\exp_{(x,0)} : N_xX \oplus T_xX \to T_xM$ is identity

($v \in N_xX$)

Because : $d\exp_{(x,0)} v = \frac{d}{dt} \exp(x, tv)\Big|_{t=0} = v$

$t \mapsto (x, tv)$ is a curve in NX through x in the direction v

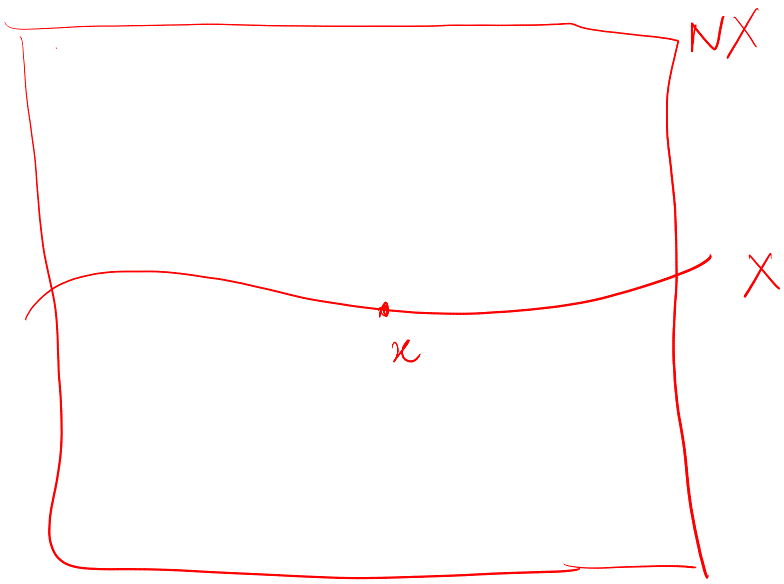
$W \in T_xX \qquad d\exp_{(x,0)}(W) = W \qquad \exp_x(0) = x$

$x \in X \subset NX$

Recall $\quad f : M \to N \qquad v \in T_mM$

$$df(v) = \frac{d}{dt} f \circ c(t)\Big|_{t=0}$$

Here $c : (-\varepsilon, \varepsilon) \to M \qquad c(0) = m$, $c'(0) = v$



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is a diffeomorphism onto its image.
- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity.

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$U_0 = \bigcup_{x \in X} U_x$

$\psi = \exp$ $\quad$ $U := \exp(U_0)$

- For ϵ small enough

$$\exp : \mathbb{D}_\epsilon(NX) \to M$$

is a diffeomorphism onto its image.

Relies on X compact

- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity. The theorem follows by the ~~inverse function theorem~~. ?

For any $x \in X$ (zero section) $\exists\, U_x \subseteq NX$ s.t. $\exp|_{U_x}$ is a diffeo

Tubular neighborhood theorem

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- Reason : For any $x \in X$, $d\exp_{(x,0)} : T_xX \oplus N_xX \to T_xM$ is identity. The theorem follows by the inverse function theorem.
- We say that $U \subset X$ is a **tubular neighborhood** of X.