Introduction to Symplectic Geometry : Lecture 4

August 27, 2021

Theorems ahead

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0,1]\}$ *is a smooth family of symplectic forms on a compact manifold* M *such that the cohomology class* $[\omega_t] \in H^2(M)$ *is t-independent. Then there exists a family of diffeomorphisms* $\rho_t : M \to M$ *such that* $\rho_t^*\omega_t = \omega_0$.

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Theorem (Neighborhood theorem in symplectic manifolds)

Let X *be a compact submanifold of a manifold* M, *and let* ω_0 , ω_1 *be closed 2-forms on* M *which are equal and non-degenerate on* $TM|X$. *Then there exist neighbourhoods* N_0 *and* N_1 *of* X *in* M *and a diffeomorphism* $\psi : N_0 \to N_1$ *which is the identity on* X *and* $\psi^* \omega_1 = \omega_0$.

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- What the theorem says : Isomorphism of normal bundles implies an isomorphism of neighborhoods.
- Darboux theorem is a particular instance of this result where $X = point$.

Vector fields and flow

- Let $v \in \mathrm{Vect}(M)$ be a vector field. The **flow** of v is a one-parameter family of diffeomorphisms $\rho_t : M \to M$ satisfying

$$\frac{d}{dt}\rho_t(m) = v(\rho_t(m)) \quad \forall m \in M, t \in \mathbb{R}.$$

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- For a time-dependent form $\omega_t \in \Omega^*(M)$, $t \in \mathbb{R}$

$$\frac{d}{dt}\rho_t^* \omega_t|_{t=\tau} = \rho_\tau^* \omega_\tau + \rho_\tau^* \frac{d}{dt}\omega_t.$$

Existence of flow for time-dependent vector fields

- Result : If M is compact and v_t, $t \in [0, 1]$ is a time-dependent vector field. Then the flow of v_t

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- Extend $\tilde{v}$ to a vector field on $\mathbb{R} \times M$ that vanishes outside a compact set, and apply the ODE result.

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- For the full result : find a smooth family $\mu_t \in \Omega^1(M)$ such that $\frac{d}{dt}\omega_t = d\mu_t$ using Hodge theory.