

Introduction to Symplectic Geometry : Lecture 4

August 27, 2021

Theorems ahead

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0, 1]\}$ is a smooth family of symplectic forms on a compact manifold M such that the cohomology class $[\omega_t] \in H^2(M)$ is t -independent. Then there exists a family of diffeomorphisms $\rho_t : M \rightarrow M$ such that $\rho_t^ \omega_t = \omega_0$.*

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Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M , and let ω_0, ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|_X$. Then there exist neighbourhoods N_0 and N_1 of X in M and a diffeomorphism $\psi : N_0 \rightarrow N_1$ which is the identity on X and $\psi^* \omega_1 = \omega_0$.

$$\forall x \in X \quad TM|_X = \bigcup_{x \in X} T_x M \supseteq TX = \bigcup_{x \in X} T_x X$$

$\omega_0|_x = \omega_1|_x$ on $T_x M$

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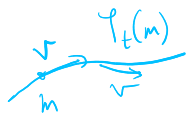
- What the theorem says : Isomorphism of normal bundles implies an isomorphism of neighborhoods.
- Darboux theorem is a particular instance of this result where $X = \text{point}$.

Vector fields and flow

- Let $v \in \text{Vect}(M)$ be a vector field. The **flow** of v is a one-parameter family of diffeomorphisms $\rho_t : M \rightarrow M$ satisfying

$$\frac{d}{dt}\rho_t(m) = v(\rho_t(m)) \quad \forall m \in M, t \in \mathbb{R}.$$

$$t \mapsto \rho_t(m)$$



Integral
curves

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$$t = \tau + h$$

- Remark : $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \frac{d}{dh}\rho_{\tau+h}^*\omega|_{h=0} = \rho_{\tau}^*L_v\omega$. (We use $\rho_{\tau+h} = \rho_h\rho_{\tau}$.)

Time-dependent vector fields

$\tilde{\rho}_t := \text{flow of } v_{t+\tau}$

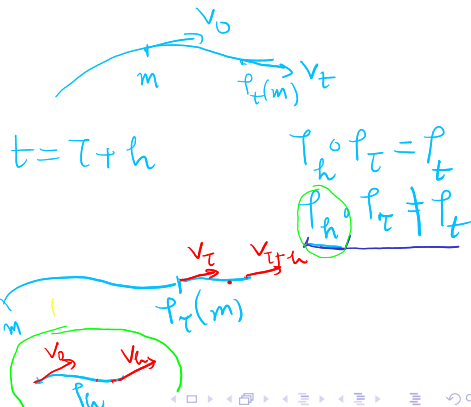
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Time ind
Time dep

$$\tilde{\rho}_h \circ \rho_\tau = \rho_{\tau+h}$$

$\rho_{\tau+h} \circ \rho_\tau^{-1}$



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- $\frac{d}{dt}\rho_t^*\omega|_{t=0} = L_{v_0}\omega$.

$$df(v) = \nabla(f) = \left. \frac{d}{dt} f(c(t)) \right|_{t=0}$$

$c : (-\varepsilon, \varepsilon) \rightarrow M$

$c'(0) = v$
 ~~$c'(t) \quad t \neq 0$~~

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- $\frac{d}{dt}\rho_t^*\omega|_{t=0} = L_{v_0}\omega$. Follows from Cartan formula.

Handwritten notes:

Cartan formula: $L_V = d \underline{i_V} + \underline{i_V} d$

holds for any ρ_t : $\frac{d}{dt} \rho_t(m) |_{t=0} = v_0$

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- $\frac{d}{dt}\rho_t^*\omega|_{t=0} = L_{v_0}\omega$. Follows from Cartan formula.
- $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} =$

$$\begin{aligned} & \downarrow \quad t = \tau + h & \rho_t &= \tilde{\rho}_h \circ \rho_\tau \\ & & \tilde{\rho}_h &: \text{Flow of } v_{t+\tau} \\ & = \frac{d}{dh} \rho_\tau^* \tilde{\rho}_h^* \omega \Big|_{h=0} \\ & = \rho_\tau^* \left(\frac{d}{dh} \tilde{\rho}_h^* \omega \Big|_{h=0} \right) = \rho_\tau^* L_{v_\tau} \omega \end{aligned}$$

Time-dependent vector fields

$$\frac{d}{dt} \rho_t^* \omega_\tau \Big|_{t=\tau} = \rho_\tau^* L_{v_\tau} \omega_\tau$$

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- $\frac{d}{dt} \rho_t^* \omega \Big|_{t=0} = L_{v_0} \omega$. Follows from Cartan formula.
- $\frac{d}{dt} \rho_t^* \omega \Big|_{t=\tau} = \frac{d}{dh} \rho_\tau^* \tilde{\rho}_h^* \omega \Big|_{h=0} = \rho_\tau^* L_{v_\tau} \omega$. (Here $\tilde{\rho}_t$ is the flow of the shifted time-dependent vector field $\tilde{v}_t := v_{t+\tau}$.)

$$\begin{aligned} \frac{d}{dt} (\rho_t^* \omega_t) \Big|_{t=\tau} \\ = \rho_\tau^* \frac{d\omega_t}{dt} \Big|_{t=\tau} \end{aligned}$$

$$\frac{d}{dt} (\rho_t^* \omega_t) \Big|_{t=\tau} = \frac{d}{dt} \rho_t^* \omega_t \Big|_{t=\tau} + \rho_\tau^* \frac{d\omega_t}{dt} \Big|_{t=\tau}$$

$\omega_t \in \Omega^*(M)$ time-dep form

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- $\frac{d}{dt}\rho_t^*\omega|_{t=0} = L_{v_0}\omega$. Follows from Cartan formula.
- $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \frac{d}{dh}\rho_\tau^*\tilde{\rho}_h^*\omega|_{h=0} = \rho_\tau^*L_{v_\tau}\omega$. (Here $\tilde{\rho}_t$ is the flow of the shifted time-dependent vector field $\tilde{v}_t := v_{t+\tau}$.)
- For a time-dependent form $\omega_t \in \Omega^*(M)$, $t \in \mathbb{R}$

$$\frac{d}{dt}\rho_t^*\omega_t|_{t=\tau} = \cancel{\rho_\tau^*\omega_\tau} + \rho_\tau^*\frac{d}{dt}\omega_t.$$

$\rho_\tau^*L_{v_\tau}\omega_\tau$

Existence of flow for time-dependent vector fields

- Result : If M is compact and $v_t, t \in [0, 1]$ is a time-dependent vector field. Then the flow of v_t

$$\rho_t : M \rightarrow M, \quad t \in [0, 1]$$

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- The proof uses the ODE result : Let $v \in \text{Vect}(M)$ be a vector field on a manifold M . For any point p there is a neighborhood $U \subset M$ and $\epsilon > 0$ such that the flow of v

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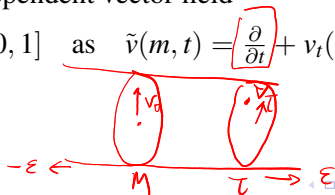
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- Think of the time-dependent vector field

$v_t \in \text{Vect}(M), t \in [0, 1]$ as $\tilde{v}(m, t) = \frac{\partial}{\partial t} + v_t(m) \in \text{Vect}(M \times [0, 1])$.



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$$v_t \in \text{Vect}(M), t \in [0, 1] \quad \text{as} \quad \tilde{v}(m, t) = \frac{\partial}{\partial t} + v_t(m) \in \text{Vect}(M \times [0, 1]).$$

$M \times [0, 1]$ If $\tilde{v}(m, t) := v_t(m)$,
will be stuck at $t=0$, hence $v_0(m)$



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- Extend \tilde{v} to a vector field on $\mathbb{R} \times M$ that vanishes outside a compact set, and apply the ODE result. AW

Moser's theorem

Theorem (Moser's theorem)

Suppose $\{\omega_t : t \in [0, 1]\}$ is a smooth family of symplectic forms on a compact manifold M such that the cohomology class $[\omega_t] \in H^2(M)$ is t -independent. Then there exists a family of diffeomorphisms $\rho_t : M \rightarrow M$ such that $\rho_t^* \omega_t = \omega_0$.

- Proof: We will make an assumption that $\omega_t = (1-t)\omega_0 + t\omega_1$.
- Since $[\omega_1] = [\omega_0]$, there exists $\mu \in \Omega^1(M)$ such that $\omega_1 - \omega_0 = d\mu$.

OBS: $[\omega_0] = [\omega_1] \in H^2(M)$, $\omega_t = (1-t)\omega_0 + t\omega_1$

$\Rightarrow [\omega_t] = [\omega_0]$

Reason: $\exists \mu \in \Omega^1(M)$

$\omega_1 - \omega_0 = d\mu$

$\omega_t = \omega_0 + t d\mu$



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$\circ \leftarrow \rho_t^* i_{v_t} \omega_t + d(i_{v_t} \omega_t)$

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- $0 = \frac{d}{dt} \rho_t^* \omega_t = \rho_t^* (L_{v_t} \omega_t + \frac{d}{dt} \omega_t) \Rightarrow L_{v_t} \omega_t + \frac{d}{dt} \omega_t = 0$
- $-L_{v_t} \omega_t = \frac{d}{dt} \omega_t = \omega_1 - \omega_0$ implies $-d(i_{v_t} \omega_t) = d\mu$.

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$$\omega_t(v_t, -) = \mu$$

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$$\frac{d\omega_t}{dt} = d\mu_t$$

Moser's theorem

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Moser's theorem

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$$m \in M$$

$$\omega_t(m) : T_m M \times T_m M \rightarrow \mathbb{R}$$

$$\mu(m) : T_m M \rightarrow \mathbb{R}$$

$$\mu(m) \in T_m^* M$$

$$\omega_t(m) : T_m M \rightarrow T_m^* M$$

ω_t is nondeg \iff isomorphism

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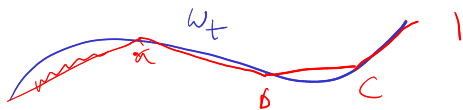
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Moser's theorem

$$\begin{aligned}
 p_a^* \omega_a &= \omega_0 \\
 p_b^* \omega_b &= \omega_0 \\
 p_1^* \omega_1 &= \omega_0
 \end{aligned}$$



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- The assumption $\omega_t = (1-t)\omega_0 + t\omega_1$ can be removed to prove the existence of ρ_1 (not a family ρ_t): Replace the smooth path by a piecewise linear path of symplectic forms with the end points ρ_0, ρ_1 .

$$\left\{ \omega \in \Omega^2(M) : \begin{aligned} d\omega &= 0 \\ [\omega] &= [\omega_0] \end{aligned} \right\}$$

vector space

(choose any topology)

non-degeneracy is
an open condition.



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- For the full result : find a smooth family $\mu_t \in \Omega^1(M)$ such that $\frac{d}{dt}\omega_t = d\mu_t$ using Hodge theory.