

Introduction to Symplectic Geometry : Lecture 3

August 23, 2021

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- The zero section of the cotangent bundle is a Lagrangian submanifold of T^*X .

Lagrangian submanifolds in the cotangent bundle

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- More rigorously : Suppose there is a family of embeddings of X

$$i : (-\epsilon, \epsilon) \times X \rightarrow T^*X, \quad i(0, x) = x,$$

such that i is continuous and the derivative of $i_t := i(t, \cdot) : X \rightarrow T^*X$ is continuous in $(-\epsilon, \epsilon) \times X$. Then, after possibly decreasing ϵ , $i_t := i(t, \cdot) : X \rightarrow T^*X$ is the image of a section of T^*X for $t \in (-\epsilon, \epsilon)$.

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$$\begin{aligned} \omega_{can}(v_i, v_j) &= (\sum_\alpha dx_\alpha \wedge d\xi_\alpha)(v_i, v_j) \\ &= dx_i(v_i) d\xi_j(v_j) - dx_j(v_j) d\xi_i(v_i) \\ &= \frac{\partial \mu_i}{\partial x_j} - \frac{\partial \mu_j}{\partial x_i}. \end{aligned}$$

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Other examples of Lagrangian submanifolds : The diagonal in a product

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Graph of a symplectomorphism

- Result : Suppose $\phi : (M, \omega) \rightarrow (M, \omega)$ is a symplectomorphism. Then the graph

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iff ϕ is a symplectomorphism.
- We have shown : ϕ is a symplectomorphism iff its graph is Lagrangian in $M \times M$.

A general neighborhood theorem in symplectic manifolds

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M , and let ω_0, ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|_X$. Then there exist neighbourhoods N_0 and N_1 of X in M and a diffeomorphism $\psi : N_0 \rightarrow N_1$ which is the identity on X and $\psi^\omega_1 = \omega_0$.*

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- What the theorem says : Isomorphism of normal bundles implies an isomorphism of neighborhoods.
- Darboux theorem is a particular instance of this result where $X = \text{point}$.
- We build up some preliminaries for the proof of the neighborhood theorem.

Vector fields and flow

- Let $v \in \text{Vect}(M)$ be a vector field. The **flow** of v is a one-parameter family of diffeomorphisms $\rho_t : M \rightarrow M$ satisfying

$$\frac{d}{dt}\rho_t(m) = v(\rho_t(m)) \quad \forall m \in M, t \in \mathbb{R}.$$

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- Remark : $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \frac{d}{dh}\rho_\tau^*\rho_h^*\omega|_{h=0} = \rho_\tau^*L_v\omega$. (We use $\rho_{\tau+h} = \rho_h\rho_\tau$.)

Time-dependent vector fields

- Let $v_t \in \text{Vect}(M)$, $t \in \mathbb{R}$ be a time-dependent vector field. The **flow** of v_t is a one-parameter family of diffeomorphisms $\rho_t : M \rightarrow M$ satisfying

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- $\frac{d}{dt}\rho_t^* \omega|_{t=\tau} = \frac{d}{dh}\rho_\tau^* \tilde{\rho}_h^* \omega|_{h=0} = \rho_\tau^* L_{v_\tau} \omega$. (Here $\tilde{\rho}_t$ is the flow of the shifted time-dependent vector field $\tilde{v}_t := v_{t+\tau}$.)

Time-dependent vector fields

For a time-dependent form $\omega_t \in \Omega^*(M)$, $t \in \mathbb{R}$

$$\frac{d}{dt} \rho_t^* \omega_t |_{t=\tau} = \rho_\tau^* \omega_\tau + \rho_\tau^* \frac{d}{dt} \omega_t.$$

Tubular neighborhood theorem

Theorem (Tubular neighborhood theorem)

Let $X \subset M$ be a compact submanifold. Then there exists

- a neighborhood $U_0 \subset NX$ of the zero section,
 - a neighborhood $U \subset M$ of X ,
 - and a diffeomorphism $\psi : U_0 \rightarrow U$ that is equal to Id_X on the zero section.
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- The normal bundle of a submanifold $X \subset M$ is

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- The proof of the theorem uses the exponential map which we recall.

The exponential map

Suppose M is equipped with a Riemannian metric.

- For any point $m \in M$, $\exp_m : T_m M \rightarrow M$ is defined so that $t \mapsto \exp_m(tv)$ is a geodesic, $\exp_m(0) = m$ and $\frac{d}{dt}(\exp_m(tv))|_{t=0} = v$.

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- For any $\epsilon \in \mathbb{R}$, the time ϵ flow is $\exp_m(\epsilon v)$.

The exponential map : Existence

- Let $S \subset M$ be a compact submanifold and let

$$\mathbb{D}(TM|S) := \{v \in T_sM : s \in S, |v| \leq 1\}$$

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- Therefore \exp is well-defined on $\mathbb{D}_\epsilon(TM|S) := \{v \in TM|S : |v| \leq \epsilon\}$ and

$$\exp : \mathbb{D}_\epsilon(TM|S) \rightarrow M$$

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- We say that $U \subset M$ is a tubular neighborhood of X .

A primitive in a tubular neighborhood

Theorem (Construction of primitive)

Let $U \subset M$ be a tubular neighborhood of a submanifold $X \subset M$. Let $i : X \rightarrow U$ be the inclusion map. Let

$$\omega \in \Omega^2(U), \quad d\omega = 0, \quad i^*\omega = 0.$$

Then there exists

$$\mu \in \Omega^1(U), \quad d\mu = \omega, \quad \mu_x = 0 \forall x \in X.$$

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- Thus $\mu := Q\omega := \int_0^1 \rho_t^* (i_{v_t} \omega)$