Introduction to Symplectic Geometry : Lecture 3

August 23, 2021

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- The zero section of the cotangent bundle is a Lagrangian submanifold of T^*X.

Lagrangian submanifolds in the cotangent bundle

Denote the zero section in T^*X by X_0.

- Question : Can we find more Lagrangian submanifolds ‘close’ to the zero section in T^*X?

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  $X_0 \subseteq T^*X$

- Claim : If $X_1 \subset T^*X$ is C^1-close to X_0 then X_1 is a section of T^*X.

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Denote the zero section in T^*X by X_0.

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- Claim : If $X_1 \subset T^*X$ is C^1-close to X_0 then X_1 is a section of T^*X.
- More rigorously : Suppose there is a family of embeddings of X

$$i : (-\epsilon, \epsilon) \times X \to T^*X, \quad i(0, x) = x,$$

$i_0(X)=X_0$

such that i is continuous and the derivative of $i_t := i(t, \cdot) : X \to T^*X$ is continuous in $(-\epsilon, \epsilon) \times X$. Then, after possibly decreasing ϵ, $i_t := i(t, \cdot) : X \to T^*X$ is the image of a section of T^*X for $t \in (-\epsilon, \epsilon)$.

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Lagrangian submanifolds in the cotangent bundle

X is compact $\Rightarrow \exists\, \varepsilon_0 : |t| < \varepsilon_0$ $d(\pi \circ i_t)_x$ is inv $\forall x \in X$. So $\pi \circ i_t : X \to X$ is a diffeo by inverse fn thm.

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  Define $\phi_t := (\pi \circ i_t)^{-1}$. Finally $\tilde{i}_t := i_t \circ \phi_t$ is a section for all t.

$$\tilde{i}_t : X \to T^*X$$

$$\tilde{i}_t(x) \in T^*_x X$$

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$X_\mu = \{(x, \mu(x)) : x \in X\} \subseteq T^*X$

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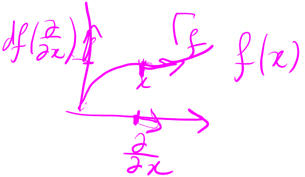
  - For $x \in X$, what is $T_{(x,\mu(x))}$ ~~T^*X~~? X_μ

$\Gamma_f \subset \mathbb{R}^2$

$T_{(x,f(x))}\Gamma \ni \frac{\partial}{\partial x} + df\left(\frac{\partial}{\partial x}\right)$



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$$\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1,\dots,n}$$

$$T_{(x,\mu(x))}X_\mu = \left\langle \frac{\partial}{\partial x_i} + d\mu\left(\frac{\partial}{\partial x_i}\right) \right\rangle_{i=1\dots n}$$

$$d\mu\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n} \frac{\partial \mu_j}{\partial x_i} \frac{\partial}{\partial \xi_j}$$

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$$\omega_{can} = \sum dx_i \wedge d\xi_i$$

X_μ is Lag $\iff$ $\forall x \; \forall i,j$ $\omega_{can}(v_i, v_j) = 0$

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Other examples of Lagrangian submanifolds : The diagonal in a product

- Let (M, ω) be a symplectic manifold. The two-form

$$\omega_{pr} \in \Omega^2(M \times M), \quad \omega_{pr} = \pi_1^*\omega - \pi_2^*\omega$$

is a symplectic form.

$\omega_{pr} = \omega \oplus -\omega$

Other examples of Lagrangian submanifolds : The diagonal in a product

$(v, w) \in T_{(m_1, m_2)} M \times M$ $\quad$ $v \in T_{m_1} M$, $w \in T_{m_2} M$

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$$T_{(m,m)}\Delta = \{(v, v) : v \in T_m M\}$$

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Graph of a symplectomorphism

- Result : Suppose $\phi : (M, \omega) \to (M, \omega)$ is a symplectomorphism. Then the graph

$$\Gamma_\phi := \{(m, \phi(m)) : m \in M\} \subset (M \times M, \omega_{pr})$$

is a Lagrangian submanifold.

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  - $\omega_{pr}((v_1, d\phi(v_1)), (v_2, d\phi(v_2))) = \omega(v_1, v_2) - \omega(d\phi(v_1), d\phi(v_2)) = 0$ for all v_1, v_2
    iff ϕ is a symplectomorphism.

$$\Updownarrow$$

$$\phi^* \omega = \omega \iff \omega(v_1, v_2) = \omega(d\phi(v_1), d\phi(v_2)) \quad \forall v_1, v_2 \in TM$$

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  - $\omega_{pr}((v_1, d\phi(v_1)), (v_2, d\phi(v_2))) = \omega(v_1, v_2) - \omega(d\phi(v_1), d\phi(v_2)) = 0$ for all v_1, v_2
    iff ϕ is a symplectomorphism.
- We have shown : ϕ is a symplectomorphism iff its graph is Lagrangian in $M \times M$.

A general neighborhood theorem in symplectic manifolds

Theorem (Neighborhood theorem in symplectic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods N_0 and N_1 of X in M and a diffeomorphism $\psi : N_0 \to N_1$ which is the identity on X and $\psi^* \omega_1 = \omega_0$.

$$TM|_X = \coprod_{x \in X} T_x M$$

$$\cup\!|$$

$$TX = \coprod_{x \in X} T_x X$$

$T_x M \quad \forall x \in X$

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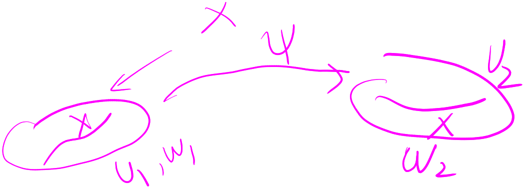
- What the theorem says : Isomorphism of normal bundles implies an isomorphism of neighborhoods.



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- Darboux theorem is a particular instance of this result where $X = point$.

A general neighborhood theorem in symplectic manifolds

Theorem (Neighborhood theorem in symplecic manifolds)

Let X be a compact submanifold of a manifold M, and let ω_0 , ω_1 be closed 2-forms on M which are equal and non-degenerate on $TM|X$. Then there exist neighbourhoods N_0 and N_1 of X in M and a diffeomorphism $\psi : N_0 \to N_1$ which is the identity on X and $\psi^*\omega_1 = \omega_0$.

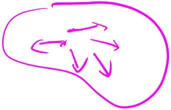
- What the theorem says : Isomorphism of normal bundles implies an isomorphism of neighborhoods.
- Darboux theorem is a particular instance of this result where $X = point$.
- We build up some preliminaries for the proof of the neighborhood theorem.

Vector fields and flow

M : manifold $v : M \to TM$ $v(m) \in T_m M$

- Let $v \in \mathrm{Vect}(M)$ be a vector field. The **flow** of v is a one-parameter family of diffeomorphisms $\rho_t : M \to M$ satisfying

$$\frac{d}{dt}\rho_t(m) = v(\rho_t(m)) \quad \forall m \in M, t \in \mathbb{R}.$$



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$\rho_t : M \to M$

- Definition(Lie derivative of forms) For a form $\omega \in \Omega^*(M)$

$$\mathcal{L}_v \omega := \frac{d}{dt}\rho_t^* \omega|_{t=0}.$$

$m \leftarrow \rho_t(m)$

$\rho_t^* \omega \in \Omega^*(M)$

$(\mathcal{L}_v \omega)(w_1, w_2) := \omega([v, w_1], w_2) - \omega(w_1, [v, w_2])$

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$$L_v = i_v d + d i_v.$$

Proof $f : M \to \mathbb{R}$

$L_v f = i_v df = v(f)$

$L_v(\alpha \wedge \beta) = L_v\alpha \wedge \beta + \alpha \wedge L_v\beta$

$(i_v d + d i_v)(\alpha \wedge \beta) = - - - -$

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$t = \tau + h$ $\qquad$ $\rho_{\tau+h} = \rho_h \circ \rho_\tau$

- Remark : $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} =$ $\frac{d}{dh}\rho_\tau^*(\rho_h^*\omega)|_{h=0}$

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- Remark : $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \frac{d}{dh}\rho_\tau^*\rho_h^*\omega|_{h=0} = \rho_\tau^* L_v\omega$. (We use $\rho_{\tau+h} = \rho_h\rho_\tau$.)

Arka

$L_{d\rho_\tau(v)}\rho_\tau^*\omega$

$\rho_t = \rho_h \circ \rho_\tau$

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Time-dependent vector fields

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- $\frac{d}{dt}\rho_t^*\omega|_{t=0} = L_{v_0}\omega$. Follows from Cartan formula.
- $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \frac{d}{dh}\rho_\tau^*\tilde{\rho}_h^*\omega|_{h=0} = \rho_\tau^* L_{v_\tau}\omega$. (Here $\tilde{\rho}_t$ is the flow of the shifted time-dependent vector field $\tilde{v}_t := v_{t+\tau}$.)