Introduction to Symplectic Geometry : Lecture 2

August 18, 2021

Symplectic form on a vector space

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- Define a subspace $V_0 = \langle v_0, w_0 \rangle$,
- and its ω-complement $V_0^\omega := \{w \in V : \omega(w, v) = 0 \forall v \in V_0\}$.

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- $\dim(V_0^\omega) = \dim(V) - 2$ and ω is symplectic on V_0^ω. The proof follows by induction.

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Claim : Let (V, Ω) be a symplectic vector space. There is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that

$$\Omega = \sum_i e_i^* \wedge f_i^*$$

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Follows from the proof of the previous claim.

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Follows from the proof of the previous claim.
Claim : Let (V,Ω) is a $2n$-dimensional symplectic vector space. Then

$$\Omega \wedge \cdots \wedge \Omega : \Lambda^n V \to \mathbb{R}$$

is non-zero.

Symplectic manifolds

Definition

A **symplectic form** ω on a manifold M is a closed 2-form

$$\omega \in \Omega^2(M), \quad d\omega = 0$$

that is a symplectic form

$$\omega_m : T_mM \times T_mM \to \mathbb{R}$$

on each tangent space T_mM for $m \in M$.

Examples of symplectic manifolds

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- Then $i_v\,\mathrm{dvol}$ is a non-vanishing area form, and hence a symplectic form.
- If v is taken to be the unit vector normal to Σ, then the area induced by the Riemannian metric coincides with $i_v\,\mathrm{dvol}$

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Let X be a smooth n-dimensional manifold. We describe a canonical symplectic form on T^*X by first describing a tautological 1-form $\alpha \in \Omega^1(T^*X)$.

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Define the canonical symplectic form as

$$\omega_{can} = -d\alpha$$

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Claim : ω is non-degenerate.

- Consider local coordinates

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- So $\alpha = \sum_i \xi_i dx_i$, and $\omega_{can} = \sum_i dx_i \wedge d\xi_i$.

Symplectic form on a cotangent bundle

A diffeomorphism $f : X_1 \to X_2$ induces a map

$$f_\# : T^*X_1 \to T^*X_2, \quad \xi \in T^*_x X_1 \mapsto \left(f^*\right)^{-1}(\xi) \in T^*_{f(x)} X_2.$$

In the homework we will show that $f_\#$ is a symplectomorphism.

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- A subspace $W \subset V$ is **Lagrangian** if $\omega|W \equiv 0$ and $\dim(W) = \frac{1}{2}\dim(V)$.

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Some submanifolds in symplectic manifolds

Let (M, ω) be a symplectic manifold.

- A submanifold $N \subset M$ is **symplectic** if for all $n \in N$, the subspace $T_nN \subset T_nM$ is symplectic. A symplectic submanifold of (M, ω) is itself a symplectic manifold.
- A submanifold $N \subset M$ is **isotropic** if for all $n \in N$, the subspace $T_nN \subset T_nM$ is isotropic.
- A submanifold $N \subset M$ is **Lagrangian** if for all $n \in N$, the subspace $T_nN \subset T_nM$ is Lagrangian.
- A submanifold $N \subset M$ is **co-isotropic** if for all $n \in N$, the subspace $T_nN \subset T_nM$ is co-isotropic.

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- In case $X = \mathbb{R}^n$, $T^*X = \mathbb{R}^{2n}$, $\omega_{can} = \sum_i dx_i \wedge d\xi_i$ globally. The sub-bundle $\{\xi_1 = \cdots = \xi_k = 0\}$ restricted to the submanifold $\{x_{k+1} = \cdots = x_n = 0\}$ is Lagrangian.

We generalize this idea.

Lagrangian submanifolds in the cotangent bundle

Definition

The **co-normal bundle** of a submanifold $S \subset X$ is

$$N^*S := \{\xi \in T_x^*X : x \in S, \xi(v) = 0 \forall v \in T_xS\}.$$

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Proof outline : On any neighborhood $U \subset X$ take local coordinates $(x_1, \ldots, x_n)$ such that

$$U \cap S = \{x_{k+1} = \cdots = x_n = 0\}.$$

For $s \in S$, $(N^*S)_s = \{\xi_1 = \cdots = \xi_k\} = 0$.

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Use the expression $\omega_{can} = \sum_i dx_i \wedge d\xi_i$.

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- Claim : If $X_1 \subset T^*X$ is C^1-close to X_0 then X_1 is a section of T^*X.
- More rigorously : Suppose there is a family of embeddings of X

$$i : (-\epsilon, \epsilon) \times X \to T^*X, \quad i(0, x) = x,$$

  such that i is continuous and the derivative of $i_t := i(t, \cdot) : X \to T^*X$ is continuous in $(-\epsilon, \epsilon) \times X$. Then, after possibly decreasing ϵ, $i_t := i(t, \cdot) : X \to T^*X$ is the image of a section of T^*X for $t \in (-\epsilon, \epsilon)$.

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  Define $\phi_t := (\pi \circ i_t)^{-1}$. Finally $\tilde{i}_t := \phi_t \circ i_t$ is a section for all t.

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