

# Introduction to Symplectic Geometry : Lecture 2

August 18, 2021

Reference : Lectures in Symplectic  
Geometry by Ana Cannas da Silva

# Symplectic form on a vector space

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Claim :  $V$  is even-dimensional.

$$\begin{aligned}\Omega : V &\rightarrow V^* \\ \ker \Omega &= \{0\}\end{aligned}$$

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- Choose  $v_0 \in V$ .

$$v_0 \neq 0$$

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- By non-degeneracy  $\exists w_0 \neq v_0$  such that  $\Omega(v_0, w_0) \neq 0$ .

$$\hookrightarrow w_0 \notin \langle v_0 \rangle$$

$$\forall \lambda \in \mathbb{R} \quad \Omega(v_0, \lambda v_0) = 0$$

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- Define a subspace  $V_0 = \langle v_0, w_0 \rangle$ ,  $\xrightarrow{\text{green}} 2\text{-d}$
- and its  $\omega$ -complement  $V_0^\omega := \{w \in V : \omega(w, v) = 0 \forall v \in V_0\}$ .  $\omega \rightarrow \Omega$

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- and its  $\omega$ -complement  $V_0^\omega := \{w \in V : \omega(w, v) = 0 \forall v \in V_0\}$ .
- $\dim(V_0^\omega) = \dim(V) - 2$  and  $\omega$  is symplectic on  $V_0^\omega$ .

*\* Figure out!*

*\*  $V = \langle v_0, w_0 \rangle \oplus V_0^\omega$  ,  $v_0, w_0 \notin V_0^\omega$*

---

*$v \in V_0^\omega \exists w \in V \quad \omega(v, w) \neq 0$*

*$\exists w' \in V_0^\omega \quad w - w' \in \langle v_0, w_0 \rangle \Rightarrow \omega(v, w) = \omega(v, w') \neq 0$*

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- and its  $\omega$ -complement  $V_0^\omega := \{w \in V : \omega(w, v) = 0 \forall v \in V_0\}$ .
- $\dim(V_0^\omega) = \dim(V) - 2$  and  $\omega$  is symplectic on  $V_0^\omega$ . The proof follows by induction.

$$\alpha v_0 + \beta w_0 \notin V_0^\omega$$

$$\text{because } \omega(\alpha v_0 + \beta w_0, \alpha w_0 - \beta v_0) \neq 0$$

$$\text{So, } V_0 \cap V_0^\omega = \{0\}$$

# Symplectic form on a vector space

Claim : Let  $(V, \Omega)$  be a symplectic vector space. There is a basis  $\{e_1, \dots, e_n, f_1, \dots, f_n\}$  such that

$$\begin{aligned} \Omega(e_i, f_i) &= 1 & \Omega &= \sum_i e_i^* \wedge f_i^* \\ \Omega(e_i, e_j) &= \Omega(f_i, f_j) = \Omega(e_i, f_j) = 0 \\ & \quad \forall i, j \end{aligned}$$

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Follows from the proof of the previous claim.

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Follows from the proof of the previous claim.

Claim : Let  $(V, \Omega)$  is a  $2n$ -dimensional symplectic vector space. Then

$$\Omega \wedge \dots \wedge \Omega : \Lambda^{2n} V \rightarrow \mathbb{R}$$

$\longleftarrow$   $n$  times  $\longrightarrow$

} HW

is non-zero.

$$\Omega : V \wedge V \rightarrow \mathbb{R}$$

## Definition

A **symplectic form**  $\omega$  on a manifold  $M$  is a closed 2-form

$$\omega \in \Omega^2(M), \quad d\omega = 0$$

that is a symplectic form

$$\omega_m : T_mM \times T_mM \rightarrow \mathbb{R}$$

on each tangent space  $T_mM$  for  $m \in M$ .

# Examples of symplectic manifolds

- Last time : the prototypical symplectic manifold is

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- Let  $d\text{vol} = dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$ .
- Let  $\Sigma \subset \mathbb{R}^3$  be an oriented surface.

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- Let  $\Sigma \subset \mathbb{R}^3$  be an oriented surface. Let  $v : \Sigma \rightarrow T\mathbb{R}^3$  be a non-vanishing vector field transverse to  $\Sigma$  everywhere.

$\forall z \in \Sigma$   $v(z) \in T_z \mathbb{R}^3$   
is transverse to  $T_z \Sigma$   
 $v(z) \neq 0$

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- Then  $i_\nu d\text{vol}$  is a non-vanishing area form, and hence a symplectic form.

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- Let  $\Sigma \subset \mathbb{R}^3$  be an oriented surface. Let  $\nu : \Sigma \rightarrow T\mathbb{R}^3$  be a non-vanishing vector field transverse to  $\Sigma$  everywhere.
- Then  $i_\nu d\text{vol}$  is a non-vanishing area form, and hence a symplectic form.
- If  $\nu$  is taken to be the unit vector normal to  $\Sigma$ , then the area induced by the Riemannian metric coincides with  $i_\nu d\text{vol}$

# Symplectic form on a cotangent bundle

Let  $X$  be a smooth  $n$ -dimensional manifold. We describe a canonical symplectic form on  $T^*X$  by first describing a tautological 1-form  $\alpha \in \Omega^1(T^*X)$ .

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- We have  $\pi_*(v) \in T_{\pi(\xi)}X$  where  $\pi : T^*M \rightarrow M$  is the projection map.

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- Define  $\alpha(v) := \xi(\pi_*(v))$ .

$$\xi \in T_{\eta(\xi)}^* X$$

$$\pi_* v \in T_{\pi(\xi)} X$$

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Define the canonical symplectic form as

$$\omega_{can} = -d\alpha$$

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Claim :  $\omega$  is non-degenerate.

- Consider local coordinates

$$(x_1, \dots, x_n) : U \rightarrow \mathbb{R}^n$$

on a neighborhood  $U \subset M$ ,

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- which will induce coordinates on the cotangent bundle

$$(x_1, \dots, x_n, \xi_1, \dots, \xi_n) : T^*U \rightarrow \mathbb{R}^n \times \mathbb{R}^n.$$

$$\xi \in T^*U \quad \xi = \sum \xi_i \overline{dx_i}(\xi)$$

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- By definition of  $\alpha$ ,  $\alpha(v) = \xi(\sum_i a_i \frac{\partial}{\partial x_i})$ .

$$\pi_* \left( \sum_i a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial \xi_i} \right) = \sum a_i \frac{\partial}{\partial x_i}$$

$$T^*U := T^*X|_U$$

$$T^*U \subset T^*X$$

open         

$$U \cup U_\alpha = X$$

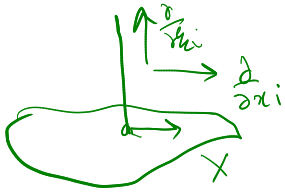
$$U \cup T^*U_\alpha = T^*X$$

$\pi: T^*X \rightarrow X$  projection

$v \in T^*X$        $\pi_* \left( \frac{\partial}{\partial \xi_i} \right) = 0$

$v = \sum a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial \xi_i}$

$\pi_* \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}$



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- By definition of  $\alpha$ ,  $\alpha(v) = \xi(\sum_i a_i \frac{\partial}{\partial x_i})$ .
- So  $\alpha = \sum_i \xi_i dx_i$ , and  $\omega_{can} = \sum_i dx_i \wedge d\xi_i$ .

$$\begin{array}{c} \parallel \\ -d\alpha \end{array}$$

# Symplectic form on a cotangent bundle

$$f: X_1 \rightarrow X_2 \quad f^*: T_{f(x)}^* X_2 \rightarrow T_x^* X_1$$

A diffeomorphism  $f: X_1 \rightarrow X_2$  induces a map

$$f_{\#}: T^* X_1 \rightarrow T^* X_2, \quad \xi \in T_x^* X_1 \mapsto (f^*)^{-1}(\xi) \in T_{f(x)}^* X_2.$$

In the homework we will show that  $f_{\#}$  is a symplectomorphism.

$$f_{\#} (T_x^* X_1) = T_{f(x)}^* X_2$$

# Some subspaces in symplectic vector spaces

Let  $(V, \Omega)$  be a symplectic vector space.

- A subspace  $W \subset V$  is **symplectic** if  $\omega|_W$  is symplectic.

# Some subspaces in symplectic vector spaces

$$\mathbb{R}^4 = \langle e_1, f_1, e_2, f_2 \rangle \quad \Omega = e_1^* \wedge f_1^* + e_2^* \wedge f_2^*$$

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- A subspace  $W \subset V$  is **symplectic** if  $\omega|_W$  is symplectic. Example :

$V = \mathbb{R}^4, \Omega = \sum_i e_i^* \wedge f_i^*$ , then  $W = \langle e_1, f_1 \rangle$  is symplectic.

- A subspace  $W \subset V$  is **isotropic** if  $\omega|_W \equiv 0$ .

$$\langle e_2, f_2 \rangle \\ V, \{0\}$$

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$\langle f_i \rangle$  etc

$V$   
↓ ?  
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 $W \subset V$  is isotropic iff  $W \subseteq W^\omega$ .
- A subspace  $W \subset V$  is **Lagrangian** if  $\omega|_W \equiv 0$  and  
 $\dim(W) = \frac{1}{2} \dim(V)$ .

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- A subspace  $W \subset V$  is **co-isotropic** if  $W^\omega \subset W$ .

$$\dim(W) \geq \frac{1}{2} \dim V$$

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- A subspace  $W \subset V$  is **co-isotropic** if  $W^\omega \subseteq W$ . Example :  
 $V = \mathbb{R}^4, \Omega = \sum_i e_i^* \wedge f_i^*$ , then  $W = \langle e_1, e_2, f_1 \rangle, V$  are co-isotropic.

$$V^\omega = \overline{\{0\}} \subseteq V$$

# Some submanifolds in symplectic manifolds

Let  $(M, \omega)$  be a symplectic manifold.

- A submanifold  $N \subset M$  is **symplectic** if for all  $n \in N$ , the subspace  $T_n N \subset T_n M$  is symplectic. A symplectic submanifold of  $(M, \omega)$  is itself a symplectic manifold.
- A submanifold  $N \subset M$  is **isotropic** if for all  $n \in N$ , the subspace  $T_n N \subset T_n M$  is isotropic.
- A submanifold  $N \subset M$  is **Lagrangian** if for all  $n \in N$ , the subspace  $T_n N \subset T_n M$  is Lagrangian.
- A submanifold  $N \subset M$  is **co-isotropic** if for all  $n \in N$ , the subspace  $T_n N \subset T_n M$  is co-isotropic.

# Lagrangian submanifolds in the cotangent bundle

$L \subset M$  Lagrangian  $\omega|_{T_e L} \equiv 0$

$$\dim(L) = \frac{1}{2} \dim(M)$$

Some Lagrangian submanifolds:  $\omega_{can} = \sum_i dx_i \wedge d\xi_i$

- The zero section  $X \subset T^*X$ .

$$TX = \left\langle \frac{\partial}{\partial x_i} \right\rangle_i$$

$$T^*X = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i} \right\rangle_i$$

# Lagrangian submanifolds in the cotangent bundle

$$\omega_{\text{can}} = \sum_i dx_i \wedge d\xi_i$$

Some Lagrangian submanifolds :

- The zero section  $X \subset T^*X$ .
- For any  $x \in X$  the cotangent fiber  $T_x^*X$ .

$$T(T_x^*X) = \left\langle \frac{\partial}{\partial \xi_i} \right\rangle_i$$

# Lagrangian submanifolds in the cotangent bundle

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- The zero section  $X \subset T^*X$ .
- For any  $x \in X$  the cotangent fiber  $T_x^*X$ .
- In case  $X = \mathbb{R}^n$ ,  $T^*X = \mathbb{R}^{2n}$ ,  $\omega_{can} = \sum_i dx_i \wedge d\xi_i$  globally.

# Lagrangian submanifolds in the cotangent bundle

$$\{\xi_i = \dots = \xi_k = 0\} = E \in T^*X$$

$$T(E|_S) = \left\langle \frac{\partial}{\partial x_i} \quad i=1 \dots k \right.$$

$$\left. \frac{\partial}{\partial \xi_i} \quad i=k+1 \dots n \right\rangle$$

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- The zero section  $X \subset T^*X$ .
- For any  $x \in X$  the cotangent fiber  $T_x^*X$ .
- In case  $X = \mathbb{R}^n$ ,  $T^*X = \mathbb{R}^{2n}$ ,  $\omega_{can} = \sum_i dx_i \wedge d\xi_i$  globally. The sub-bundle  $E = \{\xi_1 = \dots = \xi_k = 0\}$  restricted to the submanifold  $S = \{x_{k+1} = \dots = x_n = 0\}$  is Lagrangian.

We generalize this idea.

Rmk:  $\xi \in E$   
 $\Rightarrow \xi(v) = 0$   
 $\forall v \in TS.$

$E|_S$  is a  
Lag submfld  
of  $T^*X$

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The **co-normal bundle** of a submanifold  $S \subset X$  is

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Claim:  $N^*S \subset T^*X$  is a Lagrangian submanifold.

$$\begin{aligned} &\Rightarrow \\ &\dim(N^*S) \\ &= \frac{1}{2} \dim(T^*X) \end{aligned}$$

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Use the expression  $\omega_{can} = \sum_i dx_i \wedge d\xi_i$ .