# Introduction to Symplectic Geometry : Lecture 27

December 1st, 2021

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  - (smooth) for each vertex $p$ the slopes $u_1, \ldots, u_n$ of the edges form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

# Delzant's theorem

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- the set of $2n$-dimensional toric symplectic manifolds (upto equivariant symplectomorpism),
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Remark : What are all the $S^1$-toric manifolds? Answer : $(\mathbb{P}^1, c\omega_{FS})$ for any $c > 0$.

# Properties of toric manifolds

- Let $(M, \omega, T, \mu)$ be a toric manifold. Vertices of $\mu(M)$ correspond to $T$-fixed points in $M$. Fixed points in $M$ are isolated.
- For a fixed point $m_0 \in M$, there exists a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ and coordinates $(x_1, y_1, \dots, x_n, y_n)$ in a neighborhood of $m$ such that the action of $T$ is standard :

$$(z_1, \dots, z_n) \xmapsto{\exp(\theta_1, \dots, \theta_n) \in T} (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$$

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# Contrast with the case when $\dim(T) < \frac{1}{2}\dim(M)$.

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- fixed points map to the interior of $\mu(M)$,
- fixed points are not isolated,
- for an edge $e$ of the polytope $\mu^{-1}(e)$ may not be $\mathbb{P}^1$.

Homework : Construct such examples.

# Stabilizers

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$$\mu(M) = \cap_i \{x \in \mathfrak{t}^\vee : \langle x, \nu_i \rangle \leq c_i\},$$

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- If $M$ is not toric, then the stabilizers can not be read off from the moment polytope.

# Toric blow-up

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- Example: $\mathrm{Bl}_{[1:0:0]}\mathbb{P}^2$. How does the blow-up parameter $\lambda$ figure in the moment polytope?

# Variation of symplectic form in quotients

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- Thus, $\{\mu_{T_2} = \tau\}/S^1 \simeq \mathbb{P}^1$ with form $\overline{\omega}_\tau := (1 - 2\tau)\omega_{FS}$.
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- Note that $\frac{d[\overline{\omega}_\tau]}{d\tau} = -2[\omega_{FS}]$. We will relate this quantity to the first Chern class of the $S^1$-bundle $\{\mu_{T_2} = \tau\} \to \{\mu_{T_2} = \tau\}/S^1$.

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# Connections on an $S^1$-bundle

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- By an $S^1$-equivariant version of the coisotropic neighborhood theorem, there is an $S^1$-equivariant symplectomorphism

$$\phi : (Z \times (\epsilon, \epsilon), \Omega) \to \mathrm{Nbhd}(Z) \subset M$$

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# Variation of symplectic form : formula

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and

$$[\overline{\omega}_\tau] = [\overline{\omega}_0] - 2\pi\tau c_1(Z \to \overline{X}_0).$$

So, $\frac{d}{d\tau}[\omega_\tau]|_{\tau=0} = -2\pi c_1(Z)$.

# Variation of symplectic form : example

- Recall the example : Consider the action $T = (S^1)^2$ on $\mathbb{P}^2$. Denote

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- Thus, $\{\mu_{T_2} = \tau\}/S^1 \simeq \mathbb{P}^1$ with form $\overline{\omega}_\tau := (1 - 2\tau)\omega_{FS}$.
- Note that $\frac{d[\overline{\omega}_\tau]}{d\tau} = -2[\omega_{FS}]$.
- We have shown

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# Chern class of normal bundles of submanifolds

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- Let $E$ be the exceptional divisor in $\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$. The first Chern number of $NE \to E$ is $-1$.