Introduction to Symplectic Geometry : Lecture 27

December 1st, 2021

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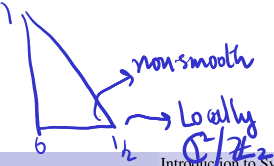
  - (smooth) for each vertex p the slopes $u_1, \ldots, u_n$ of the edges form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.



Delzant's theorem

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- the set of $2n$-dimensional toric symplectic manifolds (upto equivariant symplectomorpism),
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Remark : What are all the S^1-toric manifolds?

(M^2, ω, S^1, μ)

M

$\mu(M) = ?$

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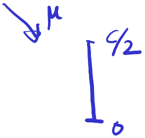
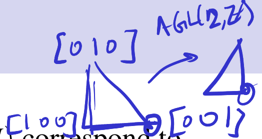
Remark : What are all the S^1-toric manifolds? Answer : $(\mathbb{P}^1, c\omega_{FS})$ for any $c > 0$.



Properties of toric manifolds



- Let (M, ω, T, μ) be a toric manifold. Vertices of $\mu(M)$ correspond to T-fixed points in M. Fixed points in M are isolated.
- For a fixed point $m_0 \in M$, there exists a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ and coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in a neighborhood of m such that the action of T is standard :

$$(z_1, \ldots, z_n) \xmapsto{\exp(\theta_1, \ldots, \theta_n) \in T} (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

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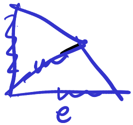

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- Let $e \subset \mu(M)$ be an edge of the polytope. Then $\mu^{-1}(e)$ is fixed by a codimension one torus $T' \subset T$. In fact $\mu^{-1}(e) \simeq \mathbb{P}^1$ with Hamiltonian action of $S^1 \simeq T/T_1$.

Contrast with the case when $\dim(T) < \frac{1}{2}\dim(M)$.

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- fixed points map to the interior of $\mu(M)$,
- fixed points are not isolated,
- for an edge e of the polytope $\mu^{-1}(e)$ may not be $\mathbb{P}^1$.

Homework : Construct such examples.

[Hint : Start with a toric mfd and restrict the action to a sub torus]

Stabilizers

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  → Hyperplane

  $$\mu(M) = \cap_i \{x \in \mathfrak{t}^\vee : \langle x, \nu_i \rangle \leq c_i\},$$

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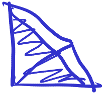
$\langle x, \nu_i \rangle = c_i$ is a facet of $\mu(M)$

eg



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$\cong S^1$

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- If M is not toric, then the stabilizers can not be read off from the moment polytope. Ex: construct such an exampl

Toric blow-up

- Recall that for ρ-blowup at 0 in $(\mathbb{C}^n, \omega_{std})$ is given by

$$\mathrm{Bl}_0^\rho(\mathbb{C}^n) := (\mathbb{C}^n \backslash B_\rho)/\sim$$

[The form on the exceptional divisor $\mathbb{P}^{n-1} \simeq E \subseteq \mathrm{Bl}^\rho_0(\mathbb{C}^n)$ is $\rho^2 \omega_{FS}$]

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The action descends to an action on the blow-up $\mathrm{Bl}_0^\rho(\mathbb{C}^n)$. What is the moment polytope?

$T = (S^1)^n$ action on $\mathrm{Bl}_0^\rho(\mathbb{C}^n)$:

* $\mathbb{C}^n \backslash \overline{B}_\rho$ is T-invariant

* Action on $\partial B_\rho / S^1$?

$\partial B_\rho \simeq S^{2n-1}$

$[z_1 : \cdots : z_n] \xmapsto{(\theta_1, \ldots, \theta_n)} [e^{i\theta_1} z_1 : \cdots : e^{i\theta_n} z_n]$

$\partial B_\rho / S^1 \simeq \{[z_1 : \cdots : z_n]\} \simeq \mathbb{P}^{n-1}$

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- -

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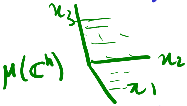
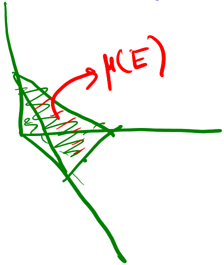
$(S^1)^n \curvearrowright \mathbb{C}^n$ $\quad \mu : \mathbb{C}^n \to \mathbb{R}^n$

$\mu(z_1 \dots z_n) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$

$\mathrm{Im}(\mu) = (\mathbb{R}_{\geq 0})^n$



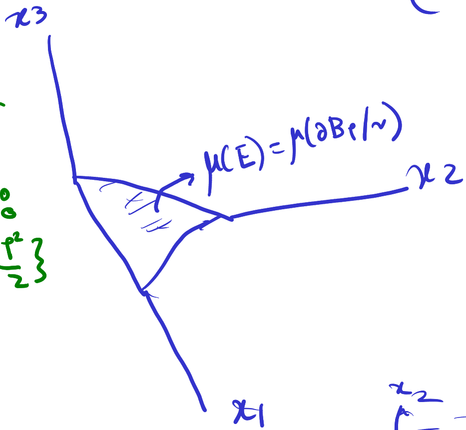
$\mu(B_\rho) = \{(x_1 \dots x_n) \in (\mathbb{R}_{\geq 0})^n : \sum_i x_i \leq \frac{\rho^2}{2}\}$



$E := \mathbb{P}^{n-1}$ is $\mu(\partial B_\rho / \sim)$

T-action descends to $Bl_0^{\rho}(\mathbb{C}^n) \simeq (\mathbb{C}^n \setminus B_\rho)/\sim$

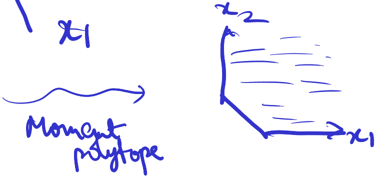
with moment polytope



$(\mathbb{R}_{\geq 0})^n \setminus \{(x_1, x_2, x_3) : \sum_i x_i = \frac{\rho^2}{2}\}$

has 4 facets

Q: $Bl_0^{\rho}(\mathbb{C}^2) \circlearrowleft (S^1)^2$



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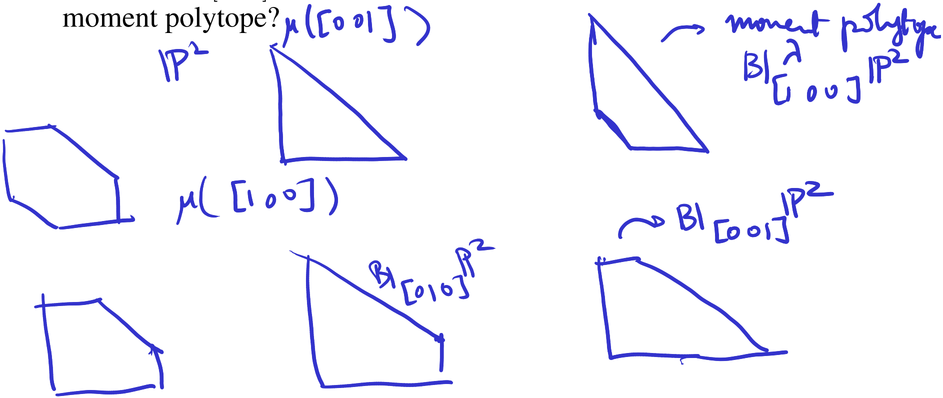

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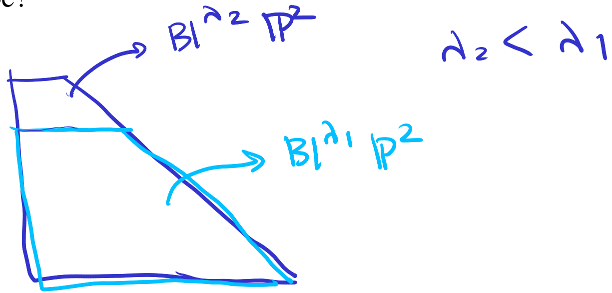

$\lambda\uparrow$, form on $E\uparrow$, chopping off a bigger piece

Variation of symplectic form in quotients

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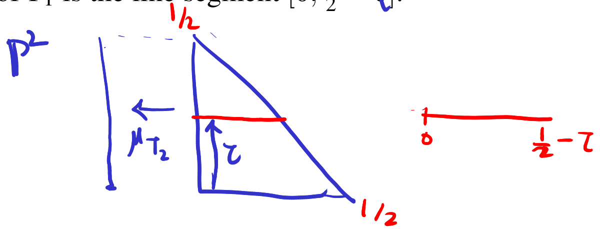

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- Thus, $\{\mu_{T_2} = \tau\}/S^1 \simeq \mathbb{P}^1$ with form $\overline{\omega}_\tau := (1 - 2\tau)\omega_{FS}$.
- Note that $\frac{d[\overline{\omega}_\tau]}{d\tau} = -2[\omega_{FS}]$.

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- Note that $\frac{d[\overline{\omega}_\tau]}{d\tau} = -2[\omega_{FS}]$. We will relate this quantity to the first Chern class of the S^1-bundle $\{\mu_{T_2} = \tau\} \to \{\mu_{T_2} = \tau\}/S^1$.

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principal S^1-bundles on X $\xleftrightarrow{\text{bijective}}$ complex line bundles on X

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- We will define the first Chern class $c_1(Z) \in H^2(X, \mathbb{Z})$, which can be proved to be equal to $c_1(L)$.

Connections on an S^1-bundle

- Given an S^1-bundle $\pi : Z \to X$, at any point $z \in Z$, the vertical subspace of T_zZ is $\ker(d\pi)$, but a horizontal subspace is not canonically defined.

$\ker d\pi_z \subseteq T_z Z$

Connections on an S^1-bundle

- Given an S^1-bundle $\pi : Z \to X$, at any point $z \in Z$, the vertical subspace of $T_z Z$ is $\ker(d\pi)$, but a horizontal subspace is not canonically defined.
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Variation of symplectic form on quotient

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$\to \mu^{-1}(\tau)/S^1$, $\mu^{-1}(0)/S^1$ are diffeomorphic

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$$\left(\overline{X}_\tau := \mu^{-1}(\tau)/S^1 , \overline{\omega}_\tau \right)$$

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Neighborhood of a level set

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$Z \subseteq M$ codim 1 hypersurface

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$$\Omega := \pi^* \overline{\omega}_0 + d(t\alpha),$$

$t \in (-\varepsilon, \varepsilon)$ coordinate

$(\mu^{-1}(0)/S^1, \overline{\omega}_0)$, with $Z = \mu^{-1}(0)$

$\overline{X}_0$

$Z \xrightarrow{\pi} (Z/S^1, \overline{\omega}_0)$

$\alpha \in \Omega^1(Z)$ connection 1-form on $Z \xrightarrow{\pi} Z/S^1 = \overline{X}_0$ (you can choose any connection 1-form)

Both are non-deg

Horiz, vertical

$t = 0 \quad \Omega = \pi^* \omega_0 + dt \wedge \alpha + t\, d\alpha \to 0$

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$\xi \in Lie(S^1) \simeq \mathbb{R}$

$i_{\xi_M}\Omega = i_{\xi_M}(dt \wedge \alpha) = -\xi dt$

$= -d(\xi t)$

$\alpha(\xi_M) = -\xi$

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$t : Z \times (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ is the moment map

- By an S^1-equivariant version of the coisotropic neighborhood theorem there is an S^1-equivariant symplectomorphism

$$\phi : (Z \times (\epsilon, \epsilon), \Omega) \to \text{Nbhd}(Z) \subset M$$

which maps $Z \times \{0\}$ to Z identically.

$\downarrow t$ $\mathbb{R}$ $\swarrow \mu$ diag commutes

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- The reduced symplectic form is

$$\overline{\omega}_\tau = \overline{\omega}_0 + \tau d\alpha,$$

τF_α where $\pi^* F_\alpha = d\alpha$

F_α is the curvature of α on the bundle $Z \to \overline{X}_0 \simeq \overline{X}_\tau$

Recall $t : Z \times (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ is the moment map for S^1-actio

$((Z \times \{\tau\})/S^1, \overline{\omega}_\tau)$ is the reduced space at level τ.

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$c_1(Z \to \overline{X}_0) = \frac{-1}{2\pi}[F_\alpha]$

and

$$[\overline{\omega}_\tau] = [\overline{\omega}_0] - 2\pi\tau c_1(Z \to \overline{X}_0).$$

So, $\frac{d}{d\tau}[\omega_\tau]|_{\tau=0} = -2\pi c_1(Z)$.

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- Recall the example : Consider the action $T = (S^1)^2$ on $\mathbb{P}^2$. Denote

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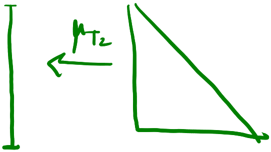

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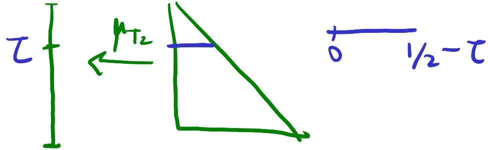

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- Thus, $\{\mu_{T_2} = \tau\}/S^1 \simeq \mathbb{P}^1$ with form $\overline{\omega}_\tau := (1 - 2\tau)\omega_{FS}$.
- Note that $\frac{d[\overline{\omega}_\tau]}{d\tau} = -2[\omega_{FS}]$.
- We have shown

$$\frac{d}{d\tau}[\overline{\omega}_\tau] = -2\pi c_1(Z),$$

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$$\int_{\mathbb{P}^1} [\omega_{FS}] = 1/4 \cdot \int_{\mathbb{P}^1} \omega_{std} = \pi$$

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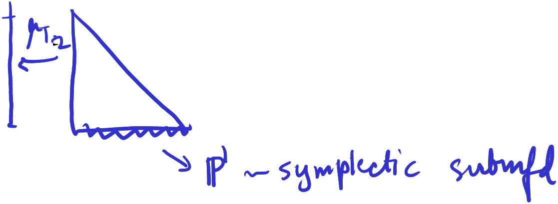
Chern class of normal bundles of submanifolds

- Consider the symplectic submanifold $X_0 := \mu_{T_2}^{-1}(0)$.



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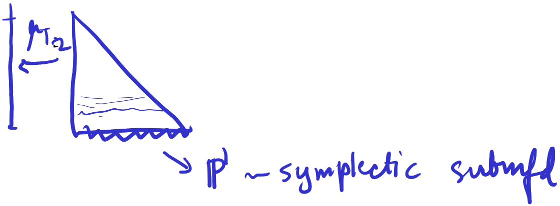
- Consider the symplectic submanifold $X_0 := \mu_{T_2}^{-1}(0)$.The unit circle bundle of NX_0 is just $Z = \mu^{-1}(\tau)$, $\tau > 0$

small



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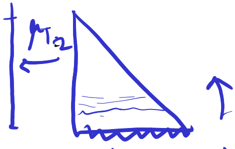

$\mathbb{P}^1$ ~ symplectic submfd

$\mathbb{P}^1 \simeq \{[z_0 : z_1 : z_2] \mid z_2 = 0\} \subseteq \mathbb{P}^2$

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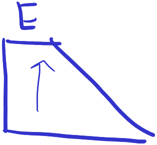
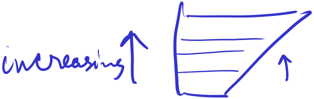


$c_1(NE) = -1$



$\frac{d}{dt}[\bar{\omega}_t] = 2[\omega_{FS}]$

Chern class of normal bundles of submanifolds

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- Let E be the exceptional divisor in $\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$. The first Chern number of $NE \to E$ is -1.