

Introduction to Symplectic Geometry : Lecture 25

November 24, 2021

Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let (X, ω, T, μ) be a compact connected T -Hamiltonian space. Then
 - ▶ the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
 - ▶ the image $\mu(X)$ is convex,
 - ▶ and $\mu(X)$ is the convex hull of $\mu(T$ -fixed points).

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- The image $\mu(X)$ is called the **moment polytope** of the Hamiltonian action.
- What's done so far : We proved convexity of $\mu(X)$ assuming connectedness of level sets of μ .
- We proved connectedness of level sets of S^1 -moment maps using Morse theory.

Detour into Morse theory (recall)

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Proof of convexity theorem, Part 2

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- Equivariant Darboux theorem : Suppose $T = (S^1)^m$ acts symplectically on (M, ω) and $p \in M$ is a fixed point. Then there are coordinates $(x_1, y_1, \dots, x_n, y_n)$ in a T -invariant neighborhood U of p such that

$$(x_1, y_1, \dots, x_n, y_n)(p) = 0$$

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- Result : Suppose T acts \mathbb{R} -linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear ω_{std} -compatible J such that the T -action is \mathbb{C} -linear.

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- Result : Suppose T acts \mathbb{R} -linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear ω_{std} -compatible J such that the T -action is \mathbb{C} -linear.
- By Homework 9, Problem 4, there is a linear change of coordinates in U to $(U, \bar{x}_1, \bar{y}_1, \dots, \bar{x}_n, \bar{y}_n)$, and weights $\mu_i \in \mathfrak{t}^\vee$ such that

$$\mu = \mu(p) + \sum_i \frac{1}{2}(\bar{x}_i^2 + \bar{y}_i^2)\mu_i.$$

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- We have shown that the level sets of an S^1 -moment map are connected.

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- The above description of a neighborhood of an S^1 -fixed point shows that any connected component in the S^1 -fixed point set is a symplectic submanifold.
- Consequently for a T -Hamiltonian action, the same is true of sets fixed by the T -action.

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Proof of the convex hull claim :

- For $F \subset \mathfrak{t}^\vee$,

$$m \in \text{Convex-Hull}(F) \Leftrightarrow \forall \xi \in \mathfrak{t} \quad \min_{x \in F} \langle x, \xi \rangle \leq \langle m, \xi \rangle \leq \max_{x \in F} \langle x, \xi \rangle.$$

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- The set of vectors with rationally independent coordinates is dense in \mathfrak{t}^\vee , so the above inequality holds for all $\xi \in \mathfrak{t}$. □

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 - ▶ (smooth) for each vertex p the slopes u_1, \dots, u_n of the edges form a \mathbb{Z} -basis of \mathbb{Z}^n .

Delzant's theorem

Delzant's theorem : There is a bijection between

- the set of $2n$ -dimensional toric symplectic manifolds (upto equivariant symplectomorphism),
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A remark about fixed points

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- The moment map in a neighborhood of m_0 is given by $\mu = \mu(m_0) + \frac{1}{2} \sum_i (x_i^2 + y_i^2) \mu_i$, where $\mu_i \in \mathfrak{t}_{\mathbb{Z}}^{\vee}$ and $\{\mu_i\}_i$ is a \mathbb{Z} -basis of $\mathfrak{t}_{\mathbb{Z}}$.

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- By changing the coordinates in \mathfrak{t} by an element in $GL(n, \mathbb{Z})$, $\{\mu_i\}_i$ is the standard basis of $\mathfrak{t}_{\mathbb{Z}}^{\vee}$.
- Therefore, a neighborhood of m_0 is T -equivariantly symplectomorphic to a neighborhood of the origin in \mathbb{C}^n with the action

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- In a toric manifold, fixed points are isolated and map to vertices of the moment polytope.