# Introduction to Symplectic Geometry : Lecture 25

November 24, 2021

# Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let $(X, \omega, T, \mu)$ be a compact connected $T$-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
  - the image $\mu(X)$ is convex,
  - and $\mu(X)$ is the convex hull of $\mu(T\text{-fixed points})$.

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  - and $\mu(X)$ is the convex hull of $\mu(T\text{-fixed points})$.
- The image $\mu(X)$ is called the **moment polytope** of the Hamiltonian action.
- What's done so far : We proved convexity of $\mu(X)$ assuming connectedness of level sets of $\mu$.
- We proved connectedness of level sets of $S^1$-moment maps using Morse theory.

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$$T_sM = T_sS \oplus H_s^+ \oplus H_s^-,$$

where $H_s^+$ resp. $H_s^-$ is the positive resp. negative eigen-space of the Hessian at $s$.

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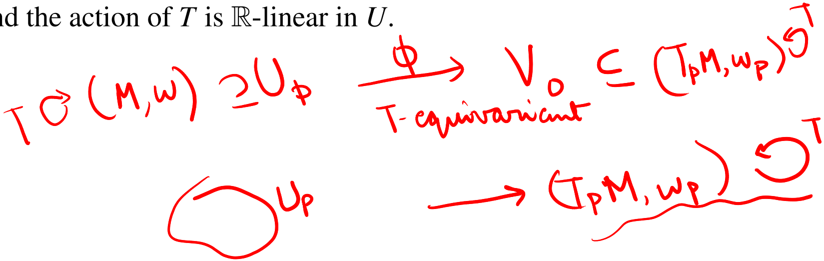
$$(x_1, y_1, \ldots, x_n, y_n)(p) = 0$$

and the action of $T$ is $\mathbb{R}$-linear in $U$.



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- Result : Suppose $T$ acts $\mathbb{R}$-linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear $\omega_{std}$-compatible $J$ such that the $T$-action is $\mathbb{C}$-linear.

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- Result : Suppose $T$ acts $\mathbb{R}$-linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear $\omega_{std}$-compatible $J$ such that the $T$-action is $\mathbb{C}$-linear.
- By Homework 9, Problem 4, there is a linear change of coordinates in $U$ to $(U, \bar{x}_1, \bar{y}_1, \dots, \bar{x}_n, \bar{y}_n)$, and weights $\mu_i \in \mathfrak{t}^\vee$ such that

$$\mu = \mu(p) + \sum_i \frac{1}{2}(\bar{x}_i^2 + \bar{y}_i^2)\mu_i.$$

$T = S^1$

$\mu_i \in \mathfrak{t}$

Rmk: $\mu_i = 0 \Rightarrow$ The $\mathbb{C}$ line $\langle x_i, y_i \rangle$ is $T$-fixed and is part of critical set

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dim (critical set) $= 2\#\{i : \mu_i = 0\}$

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- We have shown that the level sets of an $S^1$-moment map are connected.

# Observation about $T$-Hamiltonian actions

- The above description of a neighborhood of an $S^1$-fixed point shows that any connected component in the $S^1$-fixed point set is a symplectic submanifold.

$S^1 \curvearrowright (V^{2n}, \omega, J)$

$J$: compatible linear $J$

$V = \bigoplus_{i=1}^{n} V_i$

$V_k$: complex line, $\dim_{\mathbb{R}} V_k = 2$

$V_k = \langle x_k, y_k \rangle$

$V_k$ : $S_1$ invariant

$\theta \in S^1$: $x_k + i y_k \longmapsto e^{i\mu_k \theta}(x_k + i y_k)$

$\mu_k \in \mathbb{Z}$

Fixed point set $= \bigoplus_{i : \mu_i = 0} V_i \rightarrow$ symplectic subspace

# Observation about $T$-Hamiltonian actions

- The above description of a neighborhood of an $S^1$-fixed point shows that any connected component in the $S^1$-fixed point set is a symplectic submanifold.
- Consequently for a $T$-Hamiltonian action, the same is true of sets fixed by the $T$-action.

# Proof of convexity theorem, Part 2

Proof of connectedness : induction step.

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$\mu : M \to \mathfrak{t}^\vee$

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- Assume $(t_2, \ldots, t_n)$ is a regular level of $\mu_{T_{n-1}}$.

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Proof of connectedness : induction step. $T = T_1 \times T_{n-1}$

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- Assume $(t_2, \dots, t_n)$ is a regular level of $\mu_{T_{n-1}}$. $\mu_{T_{n-1}}^{-1}(t_2 \dots t_n)$ It is connected by the induction hypothesis. The quotient $\mu_{T_{n-1}}^{-1}(t_2, \dots, t_n)/T_{n-1}$ has an action of

$$T_1 \simeq \{(\theta, 1, \dots, 1)\}.$$

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$$T_1 \simeq \{(\theta, 1, \ldots, 1)\}.$$
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$T_{n-1} \to \mu^{-1}_{T_{n-1}}(t_2 \ldots t_n) \to \mu^{-1}_{T_{n-1}}(t_2 \ldots t_n)/T_{n-1}$

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- $\mu^{-1}(t_1, \ldots, t_n)$ is connected iff $\mu^{-1}_{T_1}(t_1)$ is connected, which holds by the result on $S^1$-moment maps.
- There is a problem in this proof : The action of $T_{n-1}$ may have may not be free. A slightly more sophisticated approach is required, see Atiyah p6.

# Proof of convexity theorem, Part 3

Proof of the convex hull claim :

- For $F \subset \mathfrak{t}^\vee$,

linear functional on $\mathfrak{t}^\vee$

$$m \in \text{Convex-Hull(F)} \Leftrightarrow \forall \xi \in \mathfrak{t} \ \min_{x \in F} \langle x, \xi \rangle \leq \langle m, \xi \rangle \leq \max_{x \in F} \langle x, \xi \rangle.$$

- ~~Let $\xi \in \mathfrak{t}$ be a vector with rationally independent coordinat~~es.

$$\text{Convex-Hull}(F) := \left\{ \sum t_i f_i \;\middle|\; \begin{array}{l} \sum t_i = 1 \\ t_i \in [0,1] \\ f_i \in F \end{array} \right\}$$

$(\Rightarrow)$ Easy

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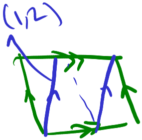
- Let $\xi \in \mathfrak{t}$ be a vector with rationally independent coordinates.

$\mathfrak{t} = \mathbb{R}^n$ $\quad \xi = (\xi_1 \dots \xi_n)$ $\quad \xi_i \in \mathbb{R}$ are $\mathbb{Q}$-indepent




Eg $n=2$ $\quad (1,2)$ is not $\mathbb{Q}$ indep

$(1,\sqrt{2})$ is $\mathbb{Q}$-indep

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$d\mu_\xi = 0$

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- The maximum and minimum points of $\mu_\xi$ are $T$-fixed points. So, for any $m \in M$

$$\min_{x \in \mathrm{Fix}_T(M)} \mu_\xi(x) \leq \mu_\xi(m) \leq \max_{x \in \mathrm{Fix}_T(M)} \mu_\xi(x).$$

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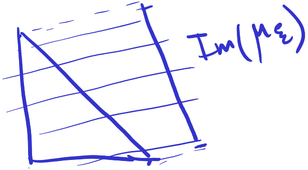
- The set of vectors with rationally independent coordinates is dense in $\mathfrak{t}^\vee$, so the above inequality holds for all $\xi \in \mathfrak{t}$. □

$T = (S^1)^2 \curvearrowright \mathbb{P}^2$

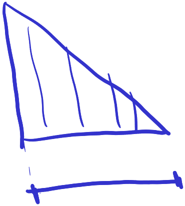
Stabilizer subgroups $= \{ Id, \{(\theta, 1)\}, \{(1, \theta)\}, \{(\theta, \theta)\}, T \}$

(*) $\xi \in t$




(A)

$\xi = (1, 0)$

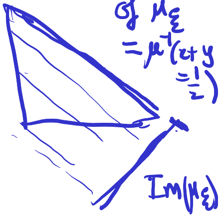


Fixed point set of $\mu_\xi$, $\xi = (1,0)$ is the $\mu^{-1}(x=0) \cong \mathbb{P}^1$

(B)

$\xi = (1, 1)$

Fixed point set of $\mu_\xi$ $= \mu^{-1}(x+y = \frac{1}{2})$



If $T_1 \subseteq T$ is a subtorus, and $T_1 \notin \mathcal{J}$

then $\mathrm{Fix}_{T_1} \subseteq M$ is exactly $\mathrm{Fix}_T$.

# Toric manifold

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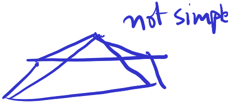
- The moment polytope of a toric manifold is a 'Delzant polytope' defined next :
- A **Delzant polytope** is a convex polytope $\Delta$ in $\mathbb{R}^n$ such that
  - (simple) there are $n$ edges meeting at a vertex,

$n=3$



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- A **Delzant polytope** is a convex polytope $\Delta$ in $\mathbb{R}^n$ such that
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# Toric manifold

- A **toric manifold** is a $T$-Hamiltonian space $(M, \omega, T, \mu)$ where $M$ is compact and connected $2n$-dimensional manifold, $T = (S^1)^n$, and the $T$-action is effective.
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  - (smooth) for each vertex $p$ the slopes $u_1, \ldots, u_n$ of the edges form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

# Delzant's theorem

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- the set of $2n$-dimensional toric symplectic manifolds (upto equivariant symplectomorpism),
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# A remark about fixed points

- Let $(M, \omega, T, \mu)$ be a toric manifold. For a vertex $p$ of $\mu(M)$, $\mu^{-1}(p)$ is a point $m_0$ in $M$ fixed by the $T$-action.

Show : $p$ is vertex of $\mu(M)$
$\Rightarrow \mu^{-1}(p)$ is fixed by the $T$-action.

$\exists\, \xi \in \mathfrak{t}$ with $\mathbb{Q}$-indep coordinates for which $p$ is a max of $\mu_\xi$

so

$\{\xi \in \mathfrak{t} : p$ is a max of $\mu_\xi\}$ is open in $\mathfrak{t}$

$\mu^{-1}(p)$ is a point $\because$ action is effective and $\dim T = n$

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- The moment map in a neighborhood of $m_0$ is given by $\mu = \mu(m_0) + \frac{1}{2}\sum_i (x_i^2 + y_i^2)\mu_i$, where $\mu_i \in \mathfrak{t}_{\mathbb{Z}}^{\vee}$ and $\{\mu_i\}_i$ is a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$.

(In particular $\mu_i \neq 0$)
$\therefore \mu^{-1}(p) =$ point

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- By changing the coordinates in $\mathfrak{t}$ by an element in in $GL(n, \mathbb{Z})$, $\{\mu_i\}_i$ is the standard basis of $\mathfrak{t}_{\mathbb{Z}}^{\vee}$.

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- Therefore, a neighborhood of $m_0$ is $T$-equivariantly symplectomorphic to a neighborhood of the origin in $\mathbb{C}^n$ with the action of $(S^1)^n$

$$(z_1, \ldots, z_n) \xrightarrow{(\theta_1, \ldots, \theta_n)} (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n).$$

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- In a toric manifold, fixed points are isolated and map to vertices of the moment polytope.