Introduction to Symplectic Geometry : Lecture 24

November 22, 2021

Moment polytopes

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  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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- Last time : We proved convexity of $\mu(X)$ assuming connectedness of level sets of μ.

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- Observation : Critical points of a Morse function are isolated.

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- Apply the same reasoning for $-f$, and we conclude that M is disconnected, which is a contradiction. $\square$

Generalization to Morse-Bott functions

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- Theorem : If a Morse-Bott function $f : M \to \mathbb{R}$ does not have critical points with index 1 or co-index 1, then its level sets are connected.

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- Equivariant Darboux theorem : Suppose $T = (S^1)^m$ acts symplectically on (M, ω) and $p \in M$ is a fixed point. Then there are coordinates $(x_1, y_1, \dots, x_n, y_n)$ in a T-invariant neighborhood U of p such that

$$(x_1, y_1, \dots, x_n, y_n)(p) = 0$$

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- Result : Suppose T acts $\mathbb{R}$-linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear ω_{std}-compatible J such that the T-action is $\mathbb{C}$-linear.
- By Homework 9, Problem 4, there is a linear change of coordinates in U to $(U, \bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n)$, and weights $\mu_i \in \mathfrak{t}^\vee$ such that

$$\mu = \mu(p) + \sum_i \frac{1}{2}(\bar{x}_i^2 + \bar{y}_i^2)\mu_i.$$

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- We have shown that the level sets of an S^1-moment map are connected.