# Introduction to Symplectic Geometry : Lecture 24

November 22, 2021

# Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let $(X, \omega, T, \mu)$ be a compact connected $T$-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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  - and $\mu(X)$ is the convex hull of $\mu(T\text{-fixed points})$.
- The image $\mu(X)$ is called the **moment polytope** of the Hamiltonian action.
- Last time : We proved convexity of $\mu(X)$ assuming connectedness of level sets of $\mu$.

We will prove $\mu^{-1}(c)$ is connected if $\mu$ is an $S^1$-moment map using Morse theory.

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- Lemma of Morse : In a neighborhood of a critical point $p$, there are coordinates $(x_1, \ldots, x_n)$ such that

$$\left( f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2. \right)$$

# Detour into Morse theory

$L_v L_w f - L_w L_v f = [v,w](f)_p = 0$

$H_p(v,w) := L_v L_{\tilde{w}} f = L_w L_{\tilde{v}} f$, $p \in Crit(f)$ $\quad v, w \in T_pM \quad \tilde{v}, \tilde{w} \in Vect(U)$

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is non-singular. $\quad H_p = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$

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Coindex(p) = n-k := # +ve of eigenvalues of $H_p$

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- Observation : Critical points of a Morse function are isolated.

Rmk $\quad$ ind$(-f,p) = n-k =$ coindex$(f,p)$

# Detour into Morse theory

- Classical Morse theory question : How does the homotopy type of

$$M_a := f^{-1}(-\infty, a] = \{ f \leq a \}$$

change as we vary $a$? Consider the example of the standing torus with $f$ being the height function.

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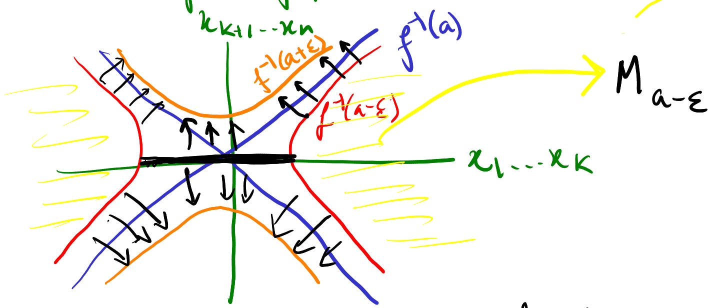
  1. If there are no critical points in $[a, b]$ then $M_a$ is diffeomorphic to $M_b$.
  2. If $a \in \mathtt{crit}(f)$ and the index of $a$ is $k$, $M_{a+\epsilon}$ is homotopy equivalent to $M_{a-\epsilon}$ with a $k$-cell attached.

$p \in M$ index $k$ critical point

$f(p) = a$

In a nbhd of $p$

$f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$




$M_{a-\varepsilon} + k\text{-disk} \simeq M_{a+\varepsilon}$

($k$-disk $= \mathbb{D}^k$, attached to $M_{a-\varepsilon}$ along $\partial \mathbb{D}^k$; $\simeq$: homotopy equiv)

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- Analogous Theorem : Let $M_{[a_0,a]} := f^{-1}([a_0, a])$.
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  $a := f(p)$ $p$ is the only critical point in $f^{-1}(a)$

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  $a := f(p)$ no other critical pt in $f^{-1}(a)$

  2. If $p \in \text{crit}(f)$ and the index of $p$ is $k$, $M_{[a_0,a+\epsilon]}$ is homotopy equivalent to $M_{[a_0,a-\epsilon]}$ with a $k$-cell attached.

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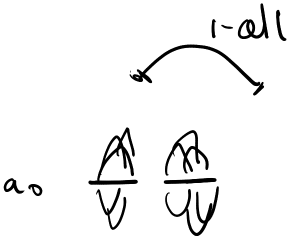


$\partial \mathbb{D}^k$ is connected if $k \geq 2$

$f^{-1}(a_0)$ is disconnected

$\Rightarrow \{f \geq a_0\}$ is also disconn with same components.

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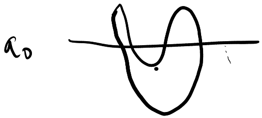
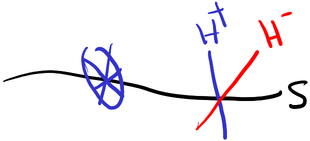
- Apply the same reasoning for $-f$, and we conclude that $M$ is disconnected, which is a contradiction. □



# Generalization to Morse-Bott functions

- A function $f : M \to \mathbb{R}$ is **Morse-Bott** if any connected critical set $S$ is a submanifold and for any $s \in S, f$ is non-degenerate in the directions normal to $S$.



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- Result : Suppose $a := f(s)$. Then, $M_{a+\epsilon}$ is homotopy equivalent to $M_{a-\epsilon}$ with an $S$-family of $k$-cells attached, where $k := \mathrm{index}(s)$.

(No other critical points in $f^{-1}(a)$ besides $S$)

Morse case : $S$ = point

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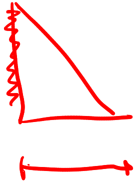
- Theorem : If a Morse-Bott function $f : M \to \mathbb{R}$ does not have critical points with index 1 or co-index 1, then its level sets are connected.

$(S^1)^2 \hookrightarrow \mathbb{P}^2$

$$[z_0 : z_1 : z_2] \overset{(\theta_1, \theta_2)}{\longmapsto} [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

$$T_1 = \{(\theta_1, 1) \in (S^1)^2\}$$

$\mathbb{P}^1$ is a critical set of $\mu_{T_1}$ ← 

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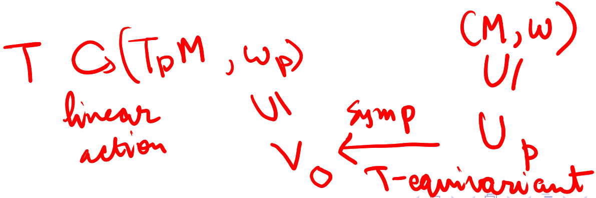
- Equivariant Darboux theorem : Suppose $T = (S^1)^m$ acts symplectically on $(M, \omega)$ and $p \in M$ is a fixed point. Then there are coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in a $T$-invariant neighborhood $U$ of $p$ such that

$$(x_1, y_1, \ldots, x_n, y_n)(p) = 0$$

and the action of $T$ is $\mathbb{R}$-linear in $U$.



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- Result : Suppose $T$ acts $\mathbb{R}$-linearly on $(\mathbb{R}^{2n}, \omega_{std})$. Then there is a linear $\omega_{std}$-compatible $J$ such that the $T$-action is $\mathbb{C}$-linear.

Pf: Same proof as contractibility of $\mathcal{J}(V,\omega)$
Start with a T-invariant metric
$\Rightarrow$ get a T-invariant J
$\Rightarrow$ T action on $(\mathbb{R}^{2n}, J)$ is $\mathbb{C}$-linear.

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- By Homework 9, Problem 4, there is a linear change of coordinates in $U$ to $(U, \bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n)$, and weights $\mu_i \in \mathfrak{t}^\vee$ such that

$$\mu = \mu(p) + \sum_i \frac{1}{2}(\bar{x}_i^2 + \bar{y}_i^2)\mu_i.$$

$T = S^1$

$\mathfrak{t}^\vee_{\mathbb{Z}} = \mathbb{Z}$

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- We have shown that the level sets of an $S^1$-moment map are connected.