Introduction to Symplectic Geometry : Lecture 23

November 17, 2021

Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let (X, ω, T, μ) be a compact connected T-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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- Last time : examples of moment polytopes for torus actions on $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^n$.
- Last time : Restricting to the action of a sub-tours $T_1 \subset T$ amounts to projecting the moment polytope by the map $i^* : \mathfrak{t}^\vee \to \mathfrak{t}_1^\vee$.

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- Performing reduction at a different level, say $\{|z_1|^2 + |z_2|^2 + |z_3|^2 = 2\}/T_1$, gives a triangle with twice the dimensions.

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Then $\mu_{T_{n-1}}^{-1}((x_2, \ldots, x_n)(p_0))$ is disconnected.

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The proof of connectedness of fibers of μ is by induction on $\dim(T)$. The base case of $T = S^1$ is proved using Morse theory.

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- Observation : Critical points of a Morse function are isolated.

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- Theorem :
  1. If there are no critical points in $[a, b]$ then M_a is diffeomorphic to M_b.
  2. If $a \in \mathrm{crit}(f)$ and the index of a is k, $M_{a+\epsilon}$ is homotopy equivalent to $M_{a-\epsilon}$ with a k-cell attached.