Introduction to Symplectic Geometry : Lecture 23

November 17, 2021

Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let (X, ω, T, μ) be a compact connected T-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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- Last time : examples of moment polytopes for torus actions on $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^n$.
- Last time : Restricting to the action of a sub-tours $T_1 \subset T$ amounts to projecting the moment polytope by the map $i^* : \mathfrak{t}^\vee \to \mathfrak{t}_1^\vee$.

Quotienting

- Consider the action of $T = (S^1)^2$ on $\mathbb{P}^2$

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Let

$$T_1 := \{(\theta, 1) \in T\}, \quad T_2 := \{(1, \theta) \in T\}.$$

Consider the quotient $\{\mu_{T_1} = c\}/S^1$. $\left(\mu_{T_1} = i^* \circ \mu_T \right)$

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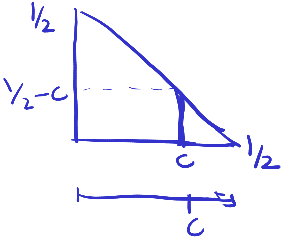

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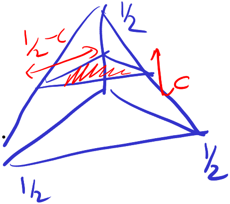

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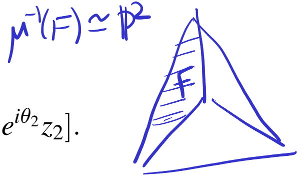

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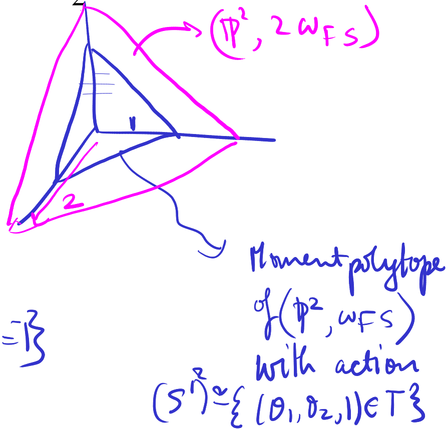
- Consider the standard action of $(S^1)^3$ on $\mathbb{C}^3$ with moment map

$$\mathbb{C}^3 \to \mathbb{R}^3, \quad (z_1, z_2, z_3) \mapsto \frac{1}{2}(|z_1|^2, |z_2|^2, |z_3|^2).$$

$\mu(\mathbb{C}^3) = (\mathbb{R}_+)^3$



$\{\sum |z_i|^2 = 1\}$

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- Performing reduction at a different level, say $\{|z_1|^2 + |z_2|^2 + |z_3|^2 = 2\}/T_1$, gives a triangle with twice the dimensions.

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$(\mathbb{R}/2\pi\mathbb{Z})^n$

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$\{(1,1),(1,-1)\} \subset \mathbb{Z}^2$ is not a $\mathbb{Z}$ basis

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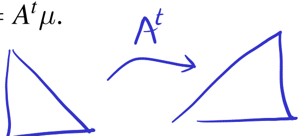

Effective T-action ⇒ $\{\mu_i \in \mathfrak{t}^*_{\mathbb{Z}}\}$ is a $\mathbb{Z}$ basis

HW9, Problem 4

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$\mu = 2$ $\qquad z \mapsto \frac{1}{2}(|z|^2 \times 2)$

$S^1 \circlearrowright \mathbb{C}$ $\qquad z \overset{\theta}{\longmapsto} e^{2i\theta} z$

not effective

$\pi \in S^1$ fixes all points in $\mathbb{C}$.

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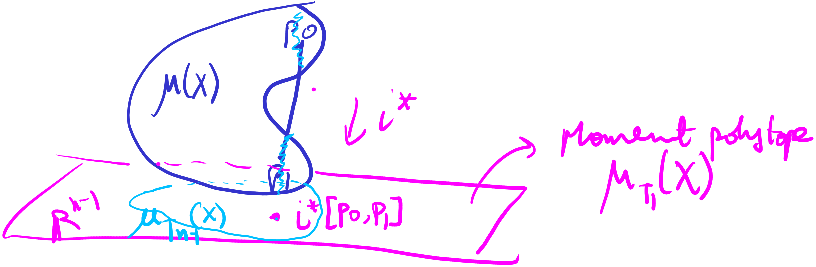
$\mu_{T_{n-1}} : X \to \mathbb{R}^{n-1}$



$\mu_{T_1}^{-1}(i^*\mu(p_0))$ is disconnected

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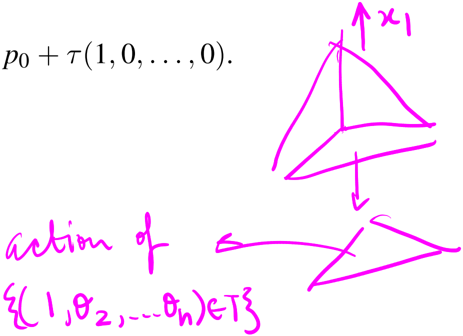

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$\tau \in \mathbb{R}$

$$i^* : \mathbb{R}^n \to \mathbb{R}^{n-1}, \quad (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n).$$

Then $\mu^{-1}_{T_{n-1}}((x_2, \ldots, x_n)(p_0))$ is disconnected.

Proof of convexity theorem, Part 1

Connectedness implies convexity.

- A non-zero vector $v \in \mathbb{Z}^n$ is **primitive** if there is no pair $t \in \mathbb{Z}_{>1}$, $v' \in \mathbb{Z}^n$ such that $v = tv'$.

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eg $(2,0)$ is not primitive
$(6,8)$

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$v = (v_1 \ldots v_n)$ is primitive $\Leftrightarrow$ $\gcd(v_1, \ldots v_n) = 1$

- Result on lattices : For any non-zero primitive vector v, there is a base change $A \in \mathrm{GL}(n, \mathbb{Z})$ such that $Av = (1, 0, \ldots, 0)$.

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\quad$ $\gcd(v_1, v_2) = 1$

$\exists \alpha, \beta \in \mathbb{Z} : \alpha v_1 + \beta v_2 = 1$

$$\begin{pmatrix} \alpha & \beta \\ -v_2 & v_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Need : $p_1 - p_0 = \tau * \text{primitive vector}$ $\quad \tau \in \mathbb{R}$

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The proof of connectedness of fibers of μ is by induction on $\dim(T)$. The base case of $T = S^1$ is proved using Morse theory.

Detour into Morse theory

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dim(M) = n

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- For a critical point p, the Hessian is symmetric. For a non-degenerate critical point, H_p has k negative eigen-values and $(n-k)$ positive eigen-values. Define the **index** of p as k.

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Refer : Morse Theory by Milnor

- Lemma of Morse : In a neighborhood of a critical point p, there are coordinates $(x_1, \ldots, x_n)$ such that

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

$H_p = 2(-1 \ldots k, +1 \ldots n-k)$

Detour into Morse theory

a is a max ind(a) = n

- Given a smooth function $f : M \to \mathbb{R}$, p is a critical point if $df(p) = 0$. The critical point p is **non-degenerate** if the Hessian

$$H_p := \nabla df_p : T_pM \otimes T_pM \to \mathbb{R}$$

  is non-singular.
- The function f is **Morse** if all critical points are non-degenerate.
- For a critical point p, the Hessian is symmetric. For a non-degenerate critical point, H_p has k negative eigen-values and $(n-k)$ positive eigen-values. Define the **index** of p as k.

Obs $a \in M$ is a min of f $\operatorname{ind}(a) = 0$ $f(x) = f(p) + x_1^2 + \cdots + x_n^2$

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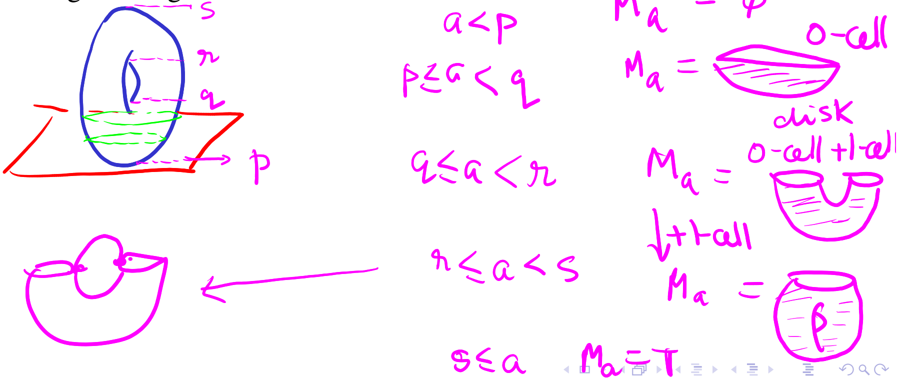
- Observation : Critical points of a Morse function are isolated.

Detour into Morse theory

- Classical Morse theory question : How does the homotopy type of

$$M_a := f^{-1}(-\infty, a] \quad = \{f \leq a\}$$

change as we vary a? Consider the example of the standing torus with f being the height function.

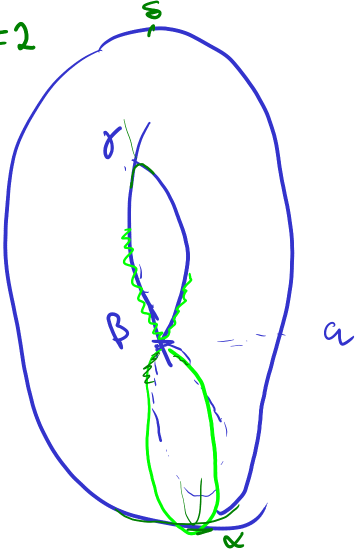


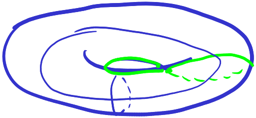
$\text{ind}(\delta) = 2$

$\text{ind}(\gamma) = 1$

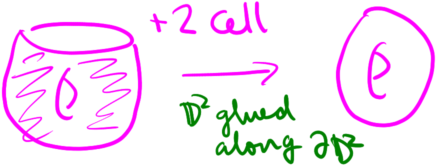
$\text{ind}(\beta) = 1$

$\text{ind}(\alpha) = 0$





$df(\beta) = df(\gamma) = 0$



Detour into Morse theory

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Detour into Morse theory

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- Theorem :
  1. If there are no critical points in $[a, b]$ then M_a is diffeomorphic to M_b.
  2. If $a \in \mathtt{crit}(f)$ and the index of a is k, $M_{a+\epsilon}$ is homotopy equivalent to $M_{a-\epsilon}$ with a k-cell attached.