Introduction to Symplectic Geometry : Lecture 22

November 15, 2021

Moment polytopes

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  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
  - the image $\mu(X)$ is convex,
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- Note : If the connected submanifold $S \subset M$ is fixed by T then $\mu(S)$ is a point.

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- Note : The area form on the unit sphere = $4 \times \omega_{FS}$.

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- For the same torus action $(\mathbb{P}^2, 2\omega_{FS})$ has moment polytope Convex-Hull$\{(0,0), (0,1), (1,0)\}$.
- For the standard action of $(S^1)^n$ on $(\mathbb{P}^n, \omega_{FS})$ the moment polytope is Convex-Hull$\{0, \frac{1}{2}e_1, \ldots, \frac{1}{2}e_n\}$.

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- In general, μ^{-1}(codimension one face of a moment polytope) is the fixed point set of an S^1-subgroup.