$n_1 < n_2 < \dots < n_k < n$

$Fl(n_1, \dots n_k) = \{ V_1 \subseteq V_2 \dots \subseteq V_k \subseteq \mathbb{C}^n \mid \dim V_i = n_i \}$

$G(n_1, n)$, $k=1$

Introduction to Symplectic Geometry : Lecture 22

November 15, 2021

Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let (M, ω, T, μ) be a compact connected T-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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- The image $\mu(X)$ is called the **moment polytope** of the Hamiltonian action.
- Note : If the connected submanifold $S \subset M$ is fixed by T then $\mu(S)$ is a point.

$\forall s \in S, t \in T \quad ts = s$

$\forall \xi \in \mathfrak{t} \quad \xi_M(s) = 0 \quad \forall s \in S$

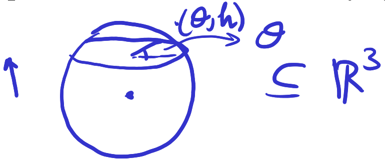
$d\mu_\xi = i_{\xi_M}\omega = 0$

μ_ξ is constt on S

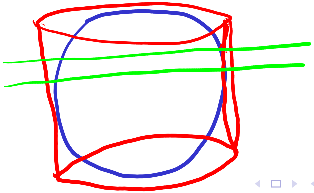
So, μ is constt on S

Examples of moment polytopes

- Example 1 : On the unit sphere $S^2 \subset \mathbb{R}^3$, $d\theta \wedge dh$ is a symplectic form on $S^2 - \{\pm 1, 0, 0\}$.



$(\theta, h) : S^2 \setminus \text{Poles} \longrightarrow S^1 \times [-1, 1]$



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$$(\theta, h) \xrightarrow{\alpha \in S^1} (\theta + \alpha, h)$$

is Hamiltonian with moment map $\mu(\theta, h) = h$.

$d\theta \wedge dh$

$d\mu = i_{\xi_M} \omega$

$\xi_M = \partial_\theta$

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- Thus the moment polytope reflects the symplectic form.
- Observe : For an (any) S^1-action, $\mu^{-1}(maximum)$ and $\mu^{-1}(minimum)$ are fixed-point sets of the S^1-action.

$d\mu_p = 0 = i_{\xi_M} \omega$

ξ is generator of Lie(S^1)

$\xi_M(p) = 0$

Examples of moment polytopes

extends continuously to a symp. form on S^2

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- Note : The area form on the unit sphere = $4 \times \omega_{FS}$.

Examples of moment polytopes

- Example (last time) : $(\mathbb{P}^2, \omega_{FS})$ with action of $T = (S^1)^2$ given by

$$[z_0 : z_1 : z_2] \xrightarrow{(\theta_1,\theta_2)} [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

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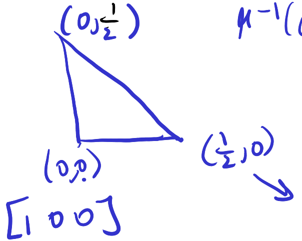

$\mu^{-1}(0, \frac{1}{2}) = \{[0\ 0\ 1]\}$
$\subseteq$
$[0\ 0\ e^{i\theta}]$

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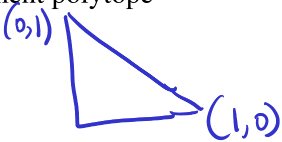

Examples of moment polytopes

Fixed points: $[1\,0 \ldots 0] \mapsto 0$, $[0 \ldots 1 \ldots 0] \mapsto \frac{1}{2}(0, \ldots 1 \ldots 0) \in \mathbb{R}^n$

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- For the same torus action $(\mathbb{P}^2, 2\omega_{FS})$ has moment polytope Convex-Hull$\{(0,0), (0,1), (1,0)\}$.
- For the standard action of $(S^1)^n$ on $(\mathbb{P}^n, \omega_{FS})$ the moment polytope is Convex-Hull$\{0, \frac{1}{2}e_1, \ldots, \frac{1}{2}e_n\}$. → n-simplex

$$[z_0 : \cdots z_n] \xmapsto{(\theta_1 \ldots \theta_n)} [z_0 : e^{i\theta_1} z_1 \ldots e^{i\theta_n} z_n]$$

$$\mu([z_0 \ldots z_n]) \in \frac{(|z_1|^2, \ldots |z_n|^2)}{2(|z_0|^2 + \ldots + |z_n|^2)}$$

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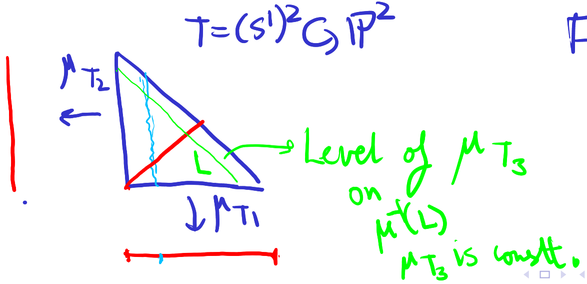

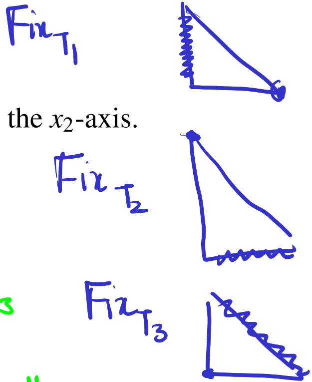

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- In general, μ^{-1}(codimension one face of a moment polytope) is the fixed point set of an S^1-subgroup.