# Introduction to Symplectic Geometry : Lecture 21

November 10, 2021

# Blowing up a point

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- There are projections $\pi : \mathrm{Bl}_0\mathbb{C}^n \to \mathbb{C}^n$, $\mathrm{pr} : \mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{CP}^{n-1}$.
- For $\lambda > 0$,

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \,\mathrm{pr}^* \,\omega_{FS}$$

is a Kähler form on $\mathrm{Bl}_0(\mathbb{C}^n)$.

# Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

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# Symplectic blowup

- Defintion of a symplectic blow-up at a point $p$ in $(M, \omega)$ : Choose a Darboux neighborhood of $p$, i.e. a symplectic embedding $i : (B_\delta, \omega_{std}) \to M$. For $\lambda < \delta$, the $\lambda$-blowup is $(M \backslash i(B_\lambda))/\sim$.
- This is a well-defined symplectic manifold $(M \backslash \overline{B}_\lambda) \cup_F (\pi^{-1}(B_\delta))$. Here $\pi : \mathrm{Bl}_0\mathbb{C}^n \to \mathbb{C}^n$ is the projection map.

# Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let $(M, \omega, T, \mu)$ be a compact connected $T$-Hamiltonian space. Then
  - the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
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- Example : $\mathbb{P}^2$ with action of $T = (S^1)^2$ given by

$$[z_0 : z_1 : z_2] \xmapsto{(\theta_1, \theta_2)} [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

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- Example : Consider the standard action $T = (S^1)^2$-action on $\mathbb{P}^2$.The restriction of this action to
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- Example : For the subgroup $T_3 := \{(\theta, \ldots, \theta) \in T\}$, the moment polytope is projected to the line $x_1 = x_2$ by $i^*$.

# Quotienting

- Using notations $T = (S^1)^2$, $T_1 := \{(\theta, 1) \in T\}$, $T_2 := \{(1, \theta) \in T\}$, consider the quotient $\{\mu_{T_1}\}/S^1$.

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- Quotienting by a subtorus amounts to taking a slice of the moment polytope.