

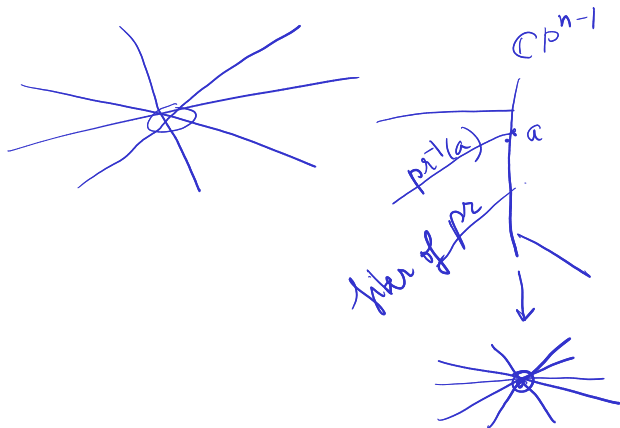
Introduction to Symplectic Geometry : Lecture 21

November 10, 2021

Blowing up a point

- As a set, blow-up of the origin in \mathbb{C}^n is

$$\text{Bl}_0(\mathbb{C}^n) := (\mathbb{C}^n \setminus \{0\}) \sqcup \text{space of lines through } 0.$$



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$$\text{Bl}_0(\mathbb{C}^n) := (\mathbb{C}^n \setminus \{0\}) \sqcup \underbrace{\text{space of lines through } 0.}_{E \simeq \mathbb{C}P^{n-1}}$$

- As a complex manifold

$$\text{Bl}_0(\mathbb{C}^n) = \{(z_1, \dots, z_n), [w_1 : \dots : w_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} : \underline{z_i w_j = z_j w_i \forall i, j}\}.$$

- There are projections $\pi : \text{Bl}_0 \mathbb{C}^n \rightarrow \mathbb{C}^n$,

$$(z, w) \mapsto z$$

Away from origin $\frac{w_i}{z_i} = \frac{w_j}{z_j}$

$$\text{Bl}_0 \mathbb{C}^n \setminus E \rightarrow \mathbb{C}^n \setminus \{0\}$$

is a biholomorphism

$$(z_1, \dots, z_n), [w_1, \dots, w_n]$$

$$\longleftarrow (z_1, \dots, z_n)$$

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- There are projections $\pi : \text{Bl}_0 \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\text{pr} : \text{Bl}_0(\mathbb{C}^n) \rightarrow \mathbb{C}\mathbb{P}^{n-1}$.

$$\{0, w\} \mapsto w$$

$$[a_1 : \dots : a_n] = a \in \mathbb{C}\mathbb{P}^{n-1},$$

$$\text{pr}^{-1}(a) = \{(\lambda a_1, \dots, \lambda a_n), [a_1, \dots, a_n] \mid \lambda \in \mathbb{C}\}$$

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$$(z, w) \mapsto z \quad (z, w) \mapsto W$$

$$\pi^* \omega_{\text{std}} + d^2 pr^* \omega_{\text{FS}} \in \mathcal{L}(Bl_0 \mathbb{C}^n)$$

$$W, V \in TBl_0 \mathbb{C}^n$$

$$\pi^* \omega_{\text{std}}(JV, JW) = \omega_{\text{std}}(d\pi(JV), d\pi(JW))$$

$$\swarrow = \omega_{\text{std}}(J(d\pi(V)), J(d\pi(W)))$$

π is holomorphic

$$= \omega_{\text{std}}(d\pi(V), d\pi(W))$$

Similarly

$d^2 pr^* \omega_{\text{FS}}$ is J -invariant
 $\therefore pr$ is holomorphic

$$\boxed{\pi^* \omega_{\text{std}} + d^2 \text{pr}^* \omega_{\text{FS}}} \in \mathcal{Z}(Bl_0 \mathbb{C}^n)$$

Kähler

Tameness? $v \in TBl_0 \mathbb{C}^n$ $v \neq 0$

$$\pi^* \omega_{\text{std}}(v, Jv)$$

$$= \omega_{\text{std}}(d\pi(v), J d\pi(v))$$

≥ 0
 $\rightarrow > 0$ if $d\pi(v) \neq 0$
 $v \in T_p(Bl_0 \mathbb{C}^n) \setminus p \notin E$

Similarly $\text{pr}^* \omega_{\text{FS}}(v, Jv) \geq 0$
 > 0 if $d\text{pr}(v) \neq 0$

$$v \neq 0 \Rightarrow d\pi(v) \neq 0$$

$\because \pi$ is a biholo
 on $Bl_0 \mathbb{C}^n \setminus E$

$$v \in T_p B|_0 \mathbb{C}^n$$

$$p \in E$$

$$\underbrace{v \in TE}$$

v is normal to E

$$pr|_E = \text{id}$$

$$v \neq 0 \quad dpr(v) \neq 0$$

$$\text{Ker } d\pi = TE$$



E

$$d\pi(v) \neq 0$$

$$v \notin \text{Ker } d\pi$$



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- As a complex manifold

$$\text{Bl}_0(\mathbb{C}^n) = \{(z_1, \dots, z_n), [w_1 : \dots : w_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z_i w_j = z_j w_i \forall i, j\}.$$

- There are projections $\pi : \text{Bl}_0 \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\text{pr} : \text{Bl}_0(\mathbb{C}^n) \rightarrow \mathbb{C}\mathbb{P}^{n-1}$.
- For $\lambda > 0$,

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \text{pr}^* \omega_{FS}$$

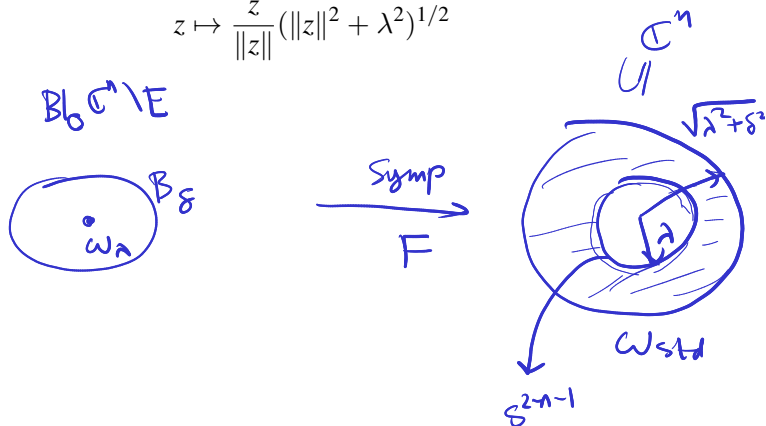
is a Kähler form on $\text{Bl}_0(\mathbb{C}^n)$.

Towards a symplectic blow-up

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \rho h^* \omega_{FS}$$

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus B_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|} (\|z\|^2 + \lambda^2)^{1/2}$$



Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus \bar{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|} (\|z\|^2 + \lambda^2)^{1/2}$$

is a symplectomorphism.

Towards a symplectic blow-up

$$\omega_\lambda = \pi^* \omega_{std} + \lambda^2 \rho^* \omega_{FS}$$

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|} (\|z\|^2 + \lambda^2)^{1/2}$$

is a symplectomorphism.

- Proof :

$$Z := \frac{z}{\|z\|} (\|z\|^2 + \lambda^2)^{1/2}$$

$$F^*(dZ \wedge d\bar{Z}) = \text{in terms of } z, \bar{z}$$

$$\begin{aligned} dz \wedge d\bar{z} &= \sum_i dz_i \wedge d\bar{z}_i & \bar{z} \cdot dz &= \sum_i \bar{z}_i dz_i \\ \rho^* \omega_{FS} &= \frac{i}{2} \partial \bar{\partial} \log \|z\|^2 & & \text{--- of } z, \bar{z} \\ &= \frac{i}{2} \frac{dz \wedge d\bar{z}}{\|z\|^2} & & \text{--- } z \cdot d\bar{z} \end{aligned}$$

$$= \frac{(\bar{z} \cdot dz) \wedge (z \cdot d\bar{z})}{\|z\|^4}$$

Towards a symplectic blow-up

$$S^{2n-1}/S^1 \simeq \mathbb{C}P^{n-1}$$

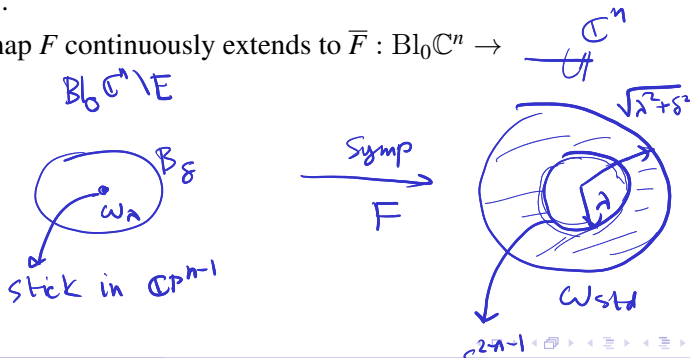
- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus \bar{B}_\lambda, \omega_{std})$ defined as

$$\subseteq \text{Bl}_0 \mathbb{C}^n \setminus E$$

$$z \mapsto \frac{z}{\|z\|} (\|z\|^2 + \lambda^2)^{1/2}$$

is a symplectomorphism.

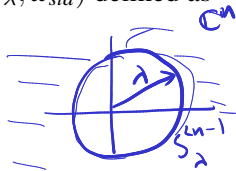
- Proof :
- The map F continuously extends to $\bar{F} : \text{Bl}_0 \mathbb{C}^n \rightarrow$



Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus \bar{B}_\lambda, \omega_{std})$ defined as

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is a symplectomorphism.

- Proof :
- The map F continuously extends to $\bar{F} : \text{Bl}_0 \mathbb{C}^n \rightarrow (\mathbb{C}^n \setminus B_\lambda) / \sim$. Here \sim mods out the boundary $\partial B_\lambda \simeq S^{2n-1}$ by the standard S^1 -action.

$$\text{on } E, \omega_\lambda = \lambda^2 \omega_{FS}$$

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \rightarrow (B_{\sqrt{\delta^2 + \lambda^2}} \setminus \bar{B}_\lambda, \omega_{std})$ defined as

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$(Bl_0 \mathbb{C}^n, \omega_\lambda)$

is

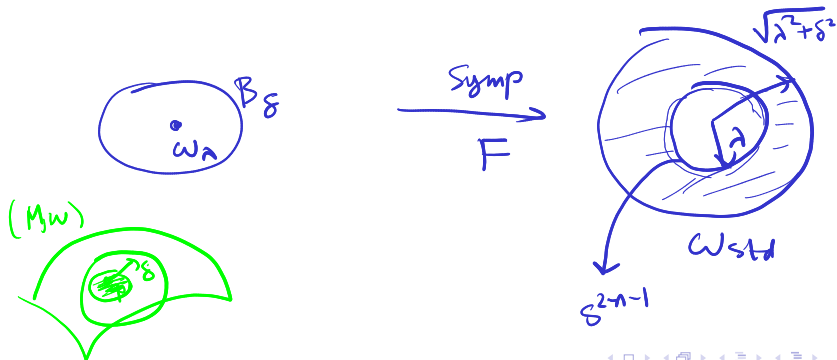
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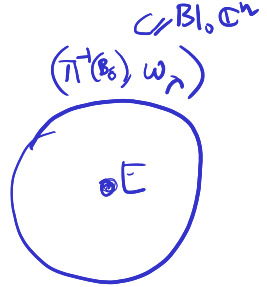
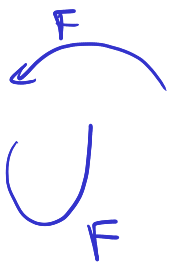
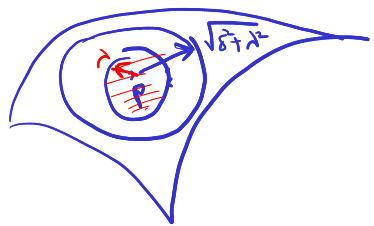
$\mathbb{C}^n \setminus B_\lambda$ is a mfd with bdry
and the bdry is $\partial B_\lambda \simeq S^{2n-1}$

Symplectic blowup

- Definition of a symplectic blow-up at a point p in (M, ω) : Choose a Darboux neighborhood of p , i.e. a symplectic embedding $i : (B_\delta, \omega_{std}) \rightarrow M$. For $\lambda < \delta$, the λ -blowup is $(M \setminus i(B_\lambda)) / \sim$.
- This is a well-defined symplectic manifold $(M \setminus \bar{B}_\lambda) \cup_F (\pi^{-1}(B_\delta))$. Here $\pi : \text{Bl}_0 \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the projection map.



$M - B_\lambda$



Nbd of E
in $B_0 \mathbb{C}^n$

Moment polytopes

- Convexity theorem (Atiyah, Guillemin-Sternberg) : Let $T = (S^1)^n$ be a torus, and let (M, ω, T, μ) be a compact connected T -Hamiltonian space. Then
 - ▶ the level set $\mu^{-1}(c)$ is connected for any $c \in \mathfrak{t}^\vee$,
 - ▶ the image $\mu(X)$ is convex,
 - ▶ and $\mu(X)$ is the convex hull of $\mu(T$ -fixed points).

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- Example : \mathbb{P}^2 with action of $T = (S^1)^2$ given by

$$[z_0 : z_1 : z_2] \xrightarrow{(\theta_1, \theta_2)} [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

has moment map

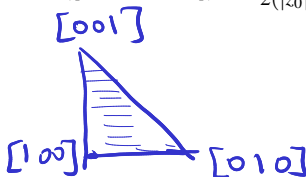
$$\frac{1}{2} \frac{(|z_1|^2, |z_2|^2)}{|z_0|^2 + |z_1|^2 + |z_2|^2}$$

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• $\mathbb{P}^n \quad (S^1)^n \curvearrowright \mathbb{P}^n \quad [z_0 : \dots : z_n] \xrightarrow{(\theta_1, \dots, \theta_n)} [z_0 : e^{i\theta_1} z_1 : \dots : e^{i\theta_n} z_n]$

$$(S^1)^n \hookrightarrow \mathbb{R}P^n \quad [z_0 \dots z_n] \xrightarrow{(0_1 \dots 0_n)} [z_0 : e^{i\theta_1} z_1 : \dots : e^{i\theta_n} z_n]$$

$$\mu([z_0 \dots z_n]) = \frac{|z_0|^2, \dots, |z_n|^2}{2(|z_0|^2 + \dots + |z_n|^2)} \in \mathbb{R}^n$$

Fixed points $\left. \begin{array}{l} [1 \ 0 \ \dots \ 0] \\ [0 \ 1 \ 0 \ \dots \ 0] \\ [0 \ 0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0] \end{array} \right\} n+1 \text{ fixed points}$

$$\begin{aligned} \mu([1 \ 0 \ 0 \ \dots \ 0]) &= (0, \dots, 0) & \mu([0 \ 1 \ 0 \ \dots \ 0]) &= (\frac{1}{2}, 0, \dots, 0) \\ \mu([0 \ \dots \ \frac{1}{2} \ 0 \ \dots \ 0]) &= (0, \dots, \frac{1}{2}, \dots, 0) \end{aligned}$$

$$\mu(X) = n\text{-simplex}$$

Moment polytope of a subgroup action

$$(M, \omega, T, \mu)$$

$$T = (S^1)^n$$

- Suppose $T_1 \subset T$ is a subgroup. The moment map of the T_1 -action is

Moment polytope of a subgroup action

- Suppose $T_1 \subset T$ is a subgroup. The moment map of the T_1 -action is $\mu_{T_1} = i^* \circ \mu$. Here $i : \mathfrak{t}_1 \rightarrow \mathfrak{t}$ is the inclusion. The dual map $i^* : \mathfrak{t}^\vee \rightarrow \mathfrak{t}_1^\vee$ is a projection assuming an inner product on \mathfrak{t} .

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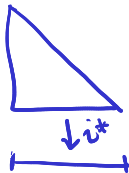
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- Example : Consider the standard action $T = (S^1)^2$ -action on \mathbb{P}^2 . The restriction of this action to
 - ▶ $T_1 := \{(\theta, 1) \in T\}$,

$$i : \mathfrak{t} \rightarrow \mathfrak{t}_1$$

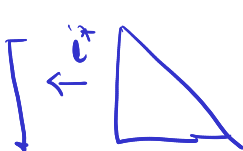
$$i^* : \mathbb{R}^2 \rightarrow \mathbb{R}$$

projects the
1st coordinate



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$$\mu_{T^2}(\mathbb{P}^2) = [0, \frac{1}{2}]$$

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projects to the moment map to the x_1 -axis and the x_2 -axis.

Moment polytope of a subgroup action

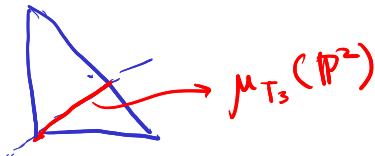
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projects to the moment map to the x_1 -axis and the x_2 -axis.

- Example : For the subgroup $T_3 := \{(\theta, \dots, \theta) \in T\}$, the moment polytope is projected to the line $x_1 = x_2$ by i^* . $T_3 \ni \theta \mapsto (\theta, \theta) \in T$



Quotienting

- Using notations $T = (S^1)^2$, $T_1 := \{(\theta, 1) \in T\}$, $T_2 := \{(1, \theta) \in T\}$, consider the quotient $\{\mu_{T_1} = c\} / S^1$.

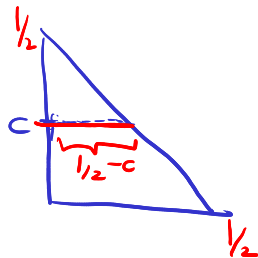
$$T_2 \curvearrowright \{\mu_{T_1} = c\} / S^1$$

↓

What is the moment
polytope of this action?

↓

.



Quotienting

- Using notations $T = (S^1)^2$, $T_1 := \{(\theta, 1) \in T\}$, $T_2 := \{(1, \theta) \in T\}$, consider the quotient $\{\mu_{T_1}\}/S^1$. The moment map for the residual action of T_2 is the line segment $[0, \frac{1}{2} - c]$.

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- Quotienting by a subtorus amounts to taking a slice of the moment polytope.

Next Class : Friday
9:30 am