Introduction to Symplectic Geometry : Lecture 20

October 25, 2021

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- $N_J \equiv 0$ iff $[T_{1,0},T_{1,0}] \subset T_{1,0}$.

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- This form on $\mathbb{P}^n$ is called the Fubini-Study form. Homework : Show that the forom on the symplectic quotient $\{\|z\|^2 = 1\}/S^1$ is equal to the Fubini-Study form.

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- Non-singular projective varieties are symplectic manifolds.

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- For $\lambda > 0$,

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \, \mathrm{pr}^* \, \omega_{FS}$$

is a Kähler form on $\mathrm{Bl}_0(\mathbb{C}^n)$.

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

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- Proof :