Introduction to Symplectic Geometry : Lecture 20

October 25, 2021

Integrability

- Dhivya's presentation : Let (M, J) be almost complex. It is integrable if and only if $[T_{1,0}, T_{1,0}] \subset T_{1,0}$.

Integrability

- Dhivya's presentation : Let (M,J) be complex. It is integrable if and only if $[T_{1,0},T_{1,0}] \subset T_{1,0}$. Equivalently $[T_{0,1},T_{0,1}] \subset T_{0,1}$.

Integrability

- Dhivya's presentation : Let (M, J) be complex. It is integrable if and only if $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Equivalently $[T_{0,1}, T_{0,1}] \subset T_{0,1}$.
- Nijenhius tensor $N_J \in \Gamma(T^*M \otimes T^*M)$ on an almost complex manifold is defined as

$$(v, w) \mapsto [v, w] - [Jv, Jw] - J[v, Jw] - J[Jv, w].$$

Q : $(v, w) \mapsto [v, w]$ tensor?

$f \in C^\infty(M)$ $\quad [fv, w] = f[v, w] \pm w(f)v$

Integrability

- Dhivya's presentation : Let (M, J) be complex. It is integrable if and only if $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Equivalently $[T_{0,1}, T_{0,1}] \subset T_{0,1}$.
- Nijenhius tensor $N_J \in \Gamma(T^*M \otimes T^*M)$ on an almost complex manifold is defined as

$$(v, w) \mapsto [v, w] - [Jv, Jw] - J[v, Jw] - J[Jv, w].$$

Ex : Show that N_J is a tensor

$$\Updownarrow$$

$\forall f, g \in C^\infty(M, \mathbb{R})$

$$N_J(fv, gw) = fg\, N_J(v, w)$$

Integrability

- Dhivya's presentation : Let (M, J) be complex. It is integrable if and only if $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Equivalently $[T_{0,1}, T_{0,1}] \subset T_{0,1}$.
- Nijenhius tensor $N_J \in \Gamma(T^*M \otimes T^*M)$ on an almost complex manifold is defined as

$$(v, w) \mapsto [v, w] - [Jv, Jw] - J[v, Jw] - J[Jv, w].$$

- $N_J \equiv 0$ iff $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. $\Longleftrightarrow$ J is integrable.

$\Longleftarrow$ $\quad X, Y \in \mathrm{Vect}(M)$

$X_{1,0} := \frac{1}{2}(X - JX \otimes i)$, $Y_{1,0} = \cdots\cdots$

$\exists Z \in \mathrm{Vect}(M) \quad Z_{1,0} = [X_{1,0}, Y_{1,0}]$

$(\Leftarrow)$ $X, Y \in \mathrm{Vect}(M)$

$$X_{1,0} := \tfrac{1}{2}(X - JX \otimes i) \,, \quad Y_{1,0} = \cdots\cdots$$

$\exists Z \in \mathrm{Vect}(M)$ $\quad Z_{1,0} = [X_{1,0}, Y_{1,0}]$

$$[X_{1,0}, Y_{1,0}] = \frac{1}{4}[X - JX\otimes i, Y - JY \otimes i]$$

$$= \frac{1}{2}(Z - JZ \otimes i)$$

$$Z = \frac{1}{2}[X,Y] - \frac{1}{2}[JX,JY] \text{ — } ①$$

$$-JZ = -\frac{1}{2}[JX,Y] - \frac{1}{2}[X,JY] \text{ — } ②$$

$-J$(RHS of ①) $=$ Rhs of ② $\Rightarrow N_J(X,Y) = 0$

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k})_{j,k}$ is positive definite.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $\left(\frac{\partial^2 f}{\underline{\partial z_j \partial \bar{z}_k}}\right)_{j,k}$ is positive definite.

$$h_{jk} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}$$

- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial} f$ is a Kähler form, and f is called a local Kähler potential.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial}f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$,

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial} f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$,

↓

$\omega = \overline{\omega}$

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial}f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. $\square$

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial} f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. □
- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial}f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. ☐
- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form $\omega = \frac{i}{2}\sum_i dz_i \wedge d\bar{z}_i$.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\underline{\partial z_j \partial z_k}})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial} f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. $\square$
- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form $\omega = \frac{i}{2}\sum_i dz_i \wedge d\bar{z}_i$.
- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial}f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. $\square$
- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form $\omega = \frac{i}{2}\sum_i dz_i \wedge d\bar{z}_i$.
- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check). The forms on all the charts patch.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.
- Claim : Then $\omega := \frac{i}{2}\partial\overline{\partial} f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$, $\partial\omega = \overline{\partial}\omega = 0$. Positive definiteness is given. ☐
- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form $\omega = \frac{i}{2}\sum_i dz_i \wedge d\bar{z}_i$.
- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check). The forms on all the charts patch.

Kähler form on $\mathbb{P}^n$

- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $\underline{f = \log(1 + \sum_{j\neq i} |z_j|^2)}$ is a local Kähler potential (check).

Affine charts

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \qquad U_i := \{z_i \neq 0\} \longrightarrow \mathbb{C}^n$$

$$(U_i, \omega_i)$$

$$[z_0 : \cdots : z_n] \mapsto \frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots \underset{\text{skip } z_i}{\downarrow} \frac{z_n}{z_i}$$

$$\left[\frac{z_0}{z_i} : \cdots : 1 : \cdots \frac{z_n}{z_i}\right]$$

$$[1 : z_1 \cdots z_n]$$

$$\mathbb{P}^n \supseteq U_0 \xrightarrow{\phi_0} \mathbb{C}^n, \omega_0 \quad (z_1 \cdots z_n)$$

$$\cup$$

$$U_1 \xrightarrow{\phi_1} \mathbb{C}^n, \omega_1$$

$$\left[\frac{1}{z_1} : 1 : \frac{z_2}{z_1} \cdots \frac{z_n}{z_1}\right]$$

$$\phi_1 \circ \phi_0^{-1}(z_1 \cdots z_n) := \left(\frac{1}{z_1}, \frac{z_2}{z_1} \cdots \frac{z_n}{z_1}\right)$$

SHOW $(\phi_1 \circ \phi_0^{-1})^* \omega_1 = \omega_0$

Kähler form on $\mathbb{P}^n$

- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
- The forms on all the charts patch:

Kähler form on $\mathbb{P}^n$

- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
- The forms on all the charts patch:
- For a holomorphic function $\phi : U \to V$, $U, V \subset \mathbb{C}^n$, $\phi^*(\Omega^{l,m}(V)) \subset \Omega^{l,m}(U)$.

Kähler form on $\mathbb{P}^n$

- Example $\mathbb{P}^n$: On a chart $[z_0, \dots, z_i = 1, \dots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
- The forms on all the charts patch:
- For a holomorphic function $\phi : U \to V$, $U, V \subset \mathbb{C}^n$, $\boxed{\phi^*(\Omega^{l,m}(V)) \subset \Omega^{l,m}(U).}$ and $\phi^* d = d \phi^*$
- So, $\phi^*\partial = \partial\phi^*$, ~~$\overline{\partial}^*\partial = \partial\overline{\partial}^*$~~.

$$\underbrace{\phi^* \bar\partial = \bar\partial \phi^*}$$

$$\begin{tikzcd}
 & \omega \in \Omega^{l,m} \arrow[dl, "\partial"'] \arrow[dr, "\bar\partial"] & \\
\Omega^{l+1,m} & & \Omega^{l,m+1}
\end{tikzcd}$$

$$\phi^* \Omega^{1,0}(V) \subseteq \Omega^{1,0}(U)$$

Kähler form on $\mathbb{P}^n$

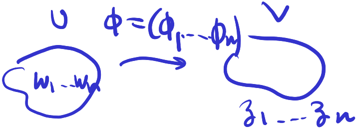
- Example $\mathbb{P}^n$: On a chart $[z_0, \ldots, z_i = 1, \ldots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
- The forms on all the charts patch:
- For a holomorphic function $\phi : U \to V$, $U, V \subset \mathbb{C}^n$, $\phi^*(\Omega^{l,m}(V)) \subset \Omega^{l,m}(U)$.
- So, $\phi^* \partial = \partial \phi^*$, $\overline{\partial}^* \partial = \partial \overline{\partial}^*$.

Kähler form on $\mathbb{P}^n$

- Example $\mathbb{P}^n$: On a chart $[z_0, \dots, z_i = 1, \dots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
- The forms on all the charts patch:
- For a holomorphic function $\phi : U \to V$, $U, V \subset \mathbb{C}^n$, $\phi^*(\Omega^{l,m}(V)) \subset \Omega^{l,m}(U)$.
- So, $\phi^* \partial = \partial \phi^*$, $\overline{\partial}^* \partial = \partial \overline{\partial}^*$.
- Identify the chart $U_0 := \{z_0 \neq 0\}$ to $\mathbb{C}^n$ by

$$[1 : z_1 : \cdots : z_n] \mapsto (z_1, \dots, z_n)$$

  do the same for any U_i.



$$\phi^* dz_i = d(z_i \circ \phi) = \sum_j \left(\frac{\partial \phi_i}{\partial w_j} dw_j + \underbrace{\frac{\partial \phi_i}{\partial \bar w_j}} d\bar w_j \right)$$

So $\phi^*(\Omega^{1,0}(V))$

$\subseteq \Omega^{1,0}(U)$.

ϕ_i holo in $(w_1 \dots w_n)$

$$\Rightarrow \frac{\partial \phi_i}{\partial \bar w_j} = 0$$

$$= \overline{\left(\frac{\partial \phi_i}{\partial \bar w_j} \right)} = 0$$

$$\phi^* d\bar z_i = d(\underbrace{\bar z_i \circ \phi}) = \sum_j \left(\boxed{\frac{\partial \bar\phi_i}{\partial w_j}} dw_j + \frac{\partial \bar\phi_i}{\partial \bar w_j} d\bar w_j \right)$$

$\phi^*(\Omega^{0,1}) \subseteq \Omega^{0,1}$ $d\bar\phi_i$

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \ldots, Z_n) \mapsto \left(z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \ldots, z_n := \frac{Z_n}{Z_1}\right).$$

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$\underline{U_0} \xrightarrow{\phi} U_1, \quad (Z_1, \dots, Z_n) \mapsto \left(z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \dots, z_n := \frac{Z_n}{Z_1}\right).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.

$$\phi^*\omega_1 = \frac{i}{2}\partial\bar{\partial}\left[\log(1+|z_0|^2+\dots+|z_n|^2)\circ\phi\right]$$

$$= \frac{i}{2}\partial\bar{\partial}\left[\log\left(1+\frac{1}{|Z_1|^2}+\frac{|Z_2|^2}{|Z_1|^2}+ + \frac{|Z_n|^2}{|Z_1|^2}\right)\right]$$

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \ldots, Z_n) \mapsto (z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \ldots, z_n := \frac{Z_n}{Z_1}).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |\frac{1}{z_1}|^2 + |\frac{z_2}{z_1}|^2 + \cdots + |\frac{z_n}{z_1}|^2)$

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \dots, Z_n) \mapsto \left(z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \dots, z_n := \frac{Z_n}{Z_1}\right).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |\frac{1}{z_1}|^2 + |\frac{z_2}{z_1}|^2 + \cdots + |\frac{z_n}{z_1}|^2)$ $=\frac{i}{2}\partial\overline{\partial}(\underbrace{\log(1 + |z_1|^2 + \cdots + |z_n|^2)}_{\omega_0} - \underbrace{\log|z_1|^2}_{}).$

$$\partial\bar\partial \log|z_1|^2 \overset{?}{=} 0$$

$$\underbrace{\log z_1}_{} + \underbrace{\log \bar z_1}_{}$$

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \ldots, Z_n) \mapsto (z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \ldots, z_n := \frac{Z_n}{Z_1}).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |\frac{1}{z_1}|^2 + |\frac{z_2}{z_1}|^2 + \cdots + |\frac{z_n}{z_1}|^2)$ $=\frac{i}{2}\partial\overline{\partial}(\log(1 + |z_1|^2 + \cdots + |z_n|^2) - \log|z_1|^2)$. Note that $\partial\overline{\partial}\log|z_1|^2 = 0$.

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \dots, Z_n) \mapsto (z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \dots, z_n := \frac{Z_n}{Z_1}).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |\frac{1}{z_1}|^2 + |\frac{z_2}{z_1}|^2 + \cdots + |\frac{z_n}{z_1}|^2)$ $=\frac{i}{2}\partial\overline{\partial}(\log(1 + |z_1|^2 + \cdots + |z_n|^2) - \log|z_1|^2)$. Note that $\partial\overline{\partial}\log|z_1|^2 = 0$. So $\phi^*\omega_1 = \omega_0$.

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \ldots, Z_n) \mapsto (z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \ldots, z_n := \frac{Z_n}{Z_1}).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1 + |\frac{1}{z_1}|^2 + |\frac{z_2}{z_1}|^2 + \cdots + |\frac{z_n}{z_1}|^2)$ $=\frac{i}{2}\partial\overline{\partial}(\log(1 + |z_1|^2 + \cdots + |z_n|^2) - \log|z_1|^2)$. Note that $\partial\overline{\partial}\log|z_1|^2 = 0$. So $\phi^*\omega_1 = \omega_0$.
- This form on $\mathbb{P}^n$ is called the Fubini-Study form.

Kähler form on $\mathbb{P}^n$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \ldots, Z_n) \mapsto \left(z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \ldots, z_n := \frac{Z_n}{Z_1}\right).$$

- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1+|z_0|^2+|z_2|^2+\cdots+|z_n|^2)$.
- Then $\phi^*\omega_1 := \frac{i}{2}\partial\overline{\partial}\log(1+|\frac{1}{z_1}|^2+|\frac{z_2}{z_1}|^2+\cdots+|\frac{z_n}{z_1}|^2)$ $=\frac{i}{2}\partial\overline{\partial}(\log(1+|z_1|^2+\cdots+|z_n|^2)-\log|z_1|^2)$. Note that $\partial\overline{\partial}\log|z_1|^2=0$. So $\phi^*\omega_1=\omega_0$.
- This form on $\mathbb{P}^n$ is called the Fubini-Study form. Homework : Show that the forom on the symplectic quotient $\underline{\{\|z\|^2=1\}/}S^1$ is equal to the Fubini-Study form.

$\{\|z\|=2\}/S^1 \rightsquigarrow$ What's the form?

Think.

Properties of Kähler manifolds

- Banyaga's theorem : Let M be a compact complex manifold. Suppose ω_0 and ω_1 are cohomologous Kähler forms. Then (M,ω_0) and (M,ω_1) are symplectomorphic.

Moser's thm hyp: $\exists\, \omega_t$ $t\in[0,1]$ $[\omega_t]$ is t-indep

ω_t is a symp form.

$\omega_t := (1-t)\omega_0 + t\omega_1$

$\omega_0 = \frac{i}{2} \sum h^0_{jk}\, dz_j \wedge d\bar{z}_k$ $\omega_1 = \frac{i}{2} \sum h^1_{jk}\, dz_j \wedge d\bar{z}_k$

Properties of Kähler manifolds

- Banyaga's theorem : Let M be a compact complex manifold. Suppose ω_0 and ω_1 are cohomologous Kähler forms. Then (M, ω_0) and (M, ω_1) are symplectomorphic.

$(h^0_{jk})_{j,k}$ $(h^1_{jk})_{j,k}$ are

positive def hermitian matrices

$$(1-t)(h^0_{j,k})_{j,k} + t(h^1_{j,k})_{j,k}$$

is also positive def $\forall t$.

ω_t is symp.

By Moser's thm (M,ω_0), (M,ω_1) are symp morphic

Properties of Kähler manifolds

- Banyaga's theorem : Let M be a compact complex manifold. Suppose ω_0 and ω_1 are cohomologous Kähler forms. Then (M, ω_0) and (M, ω_1) are symplectomorphic.
- A complex submanifold of a Kähler manifold is Kähler.

ω is a Kähler form on (M, J)

$\forall n \in N$

$T_nN \subseteq T_nM$

complex subspace

T_nN is J-closed

$\Leftarrow N \underset{i}{\subseteq} M$ is a complex submfld

$i^*\omega$

$d(i^*\omega)$ ✓

$i^*\omega \in \Omega^{1,1}$ ✓

Only need $\omega(v, Jv) > 0$

$\forall v \in TN \quad v \neq 0$

$Jv \in TN$ $\because$ N is complex submfd

Properties of Kähler manifolds

- Banyaga's theorem : Let M be a compact complex manifold. Suppose ω_0 and ω_1 are cohomologous Kähler forms. Then (M, ω_0) and (M, ω_1) are symplectomorphic.
- A complex submanifold of a Kähler manifold is Kähler.
  Enough to check that the restriction of the symplectic form is ~~non-degenerate~~. tamed

Properties of Kähler manifolds

- Banyaga's theorem : Let M be a compact complex manifold. Suppose ω_0 and ω_1 are cohomologous Kähler forms. Then (M, ω_0) and (M, ω_1) are symplectomorphic.
- A complex submanifold of a Kähler manifold is Kähler.
  Enough to check that the restriction of the symplectic form is non-degenerate.
- Non-singular projective varieties are symplectic manifolds.

Blowing up a point

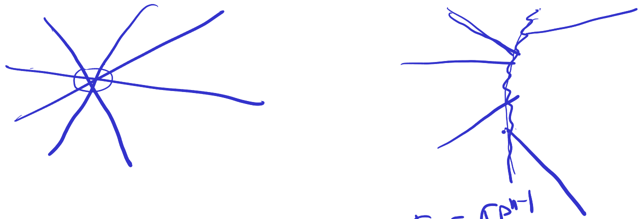
- As a set, blow-up of the origin in $\mathbb{C}^n$ is

$$\mathrm{Bl}_0(\mathbb{C}^n) := \overset{\mathbb{C}^n \setminus \{0\}}{\cancel{\mathbb{C}^n}} \sqcup \underline{\text{space of lines through } 0}.$$

is

$\mathbb{CP}^{n-1}$

$\mathbb{C}^n$



$E := \mathbb{CP}^{n-1}$

$\downarrow$

Exceptional divisor

Blowing up a point

- As a set, blow-up of the origin in $\mathbb{C}^n$ is

$$\mathrm{Bl}_0(\mathbb{C}^n) := \mathbb{C}^n \sqcup \text{space of lines through } 0.$$

- As a complex manifold

$\{0\} \times \mathbb{P}^{n-1} \subseteq \mathrm{Bl}_0\mathbb{C}^n$

Exceptional divisor

$$\mathrm{Bl}_0(\mathbb{C}^n) = \{\underline{(z_1, \ldots, z_n), [w_1 : \cdots : w_n]} \in \underline{\mathbb{C}^n \times \mathbb{P}^{n-1}} : \underline{z_i w_j = z_j w_i} \forall i, j\}.$$

- There are projections $\pi : \mathrm{Bl}_0\mathbb{C}^n \to \mathbb{C}^n$,

$\downarrow \pi$

$z = (z_1 \cdots z_n)$

$\mathrm{Bl}_0 \mathbb{C}^n \cap \{z \neq 0\} \xrightarrow{\pi} \mathbb{C}^n$ is a biholomorphism

$$\begin{tikzcd} \underline{Bl_0\mathbb{C}^n \cap \{z \neq 0\}} \arrow[r, "\pi"] & \mathbb{C}^n \setminus \{0\} \arrow[l, bend left, "\psi"] \end{tikzcd}$$

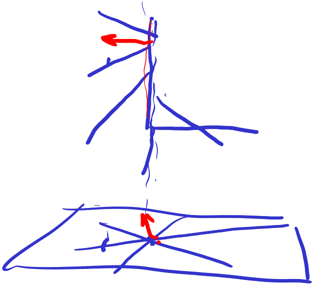
$\psi = \pi^{-1}$ on $\mathbb{C}^n \setminus \{0\}$

$$(z_1, \dots z_n), \underset{\mathbb{P}^{n-1}}{\underset{\cap}{[z_1 : \dots : z_n]}} \longleftarrow\!\!\!\shortmid \; (z_1 \dots z_n)$$

$$Bl_0\mathbb{C}^n \setminus \operatorname{Im}\psi = \{0\} \times \mathbb{C}P^{n-1}$$

$pr_2 : Bl_0\mathbb{C}^n \longrightarrow \mathbb{C}P^{n-1}$

$(z, w) \longmapsto \underline{w}$



$$\mathrm{Bl}_0\mathbb{C}^n = \{(z,w) : \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z_i w_j = z_j w_i \ \forall i,j\}$$

$(z_1, \dots z_n), [w_1 : \dots : w_n]$

$$\begin{tikzcd}
(z,w) \arrow[d, mapsto] & & \mathrm{Bl}_0\mathbb{C}^n \arrow[dl, "\pi"'] \arrow[dr, "pr"] & & (z,w) \arrow[d, mapsto] \\
z & \mathbb{C}^n & & \mathbb{C}P^{n-1} & w
\end{tikzcd}$$

$pr^{-1}(w) =$ line in $\mathbb{C}^n$ represented by w

$\mathrm{Bl}_0\mathbb{C}^n \xrightarrow{pr} \mathbb{C}P^{n-1}$ is the <u>tautological line bundle</u>

Blowing up a point

- As a set, blow-up of the origin in $\mathbb{C}^n$ is

$$\mathrm{Bl}_0(\mathbb{C}^n) := \mathbb{C}^n \sqcup \text{space of lines through } 0.$$

- As a complex manifold

$$\mathrm{Bl}_0(\mathbb{C}^n) = \{(z_1, \dots, z_n), [w_1 : \cdots : w_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z_i w_j = z_j w_i \forall i, j\}.$$

- There are projections $\pi : \mathrm{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$, $\mathrm{pr} : \mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{CP}^{n-1}$.

Blowing up a point

- As a set, blow-up of the origin in $\mathbb{C}^n$ is

$$\mathrm{Bl}_0(\mathbb{C}^n) := \mathbb{C}^n \sqcup \text{space of lines through } 0.$$

- As a complex manifold

$$\mathrm{Bl}_0(\mathbb{C}^n) = \{(z_1, \dots, z_n), [w_1 : \cdots : w_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z_i w_j = z_j w_i \forall i, j\}.$$

- There are projections $\pi : \mathrm{Bl}_0\mathbb{C}^n \to \mathbb{C}^n$, $\mathrm{pr} : \mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{CP}^{n-1}$.
- For $\lambda > 0$,

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \, \mathrm{pr}^* \, \omega_{FS}$$

is a Kähler form on $\underline{\mathrm{Bl}_0(\mathbb{C}^n)}$.

$$\begin{tikzcd}
 & \mathrm{Bl}_0\mathbb{C}^n \arrow[dl, "\pi"'] \arrow[dr, "\mathrm{Pr}"] & \\
\mathbb{C}^n & & \mathbb{CP}^{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
 & Bl_0 \mathbb{C}^n \arrow[dl, "\pi"'] \arrow[dr, "pr"] & \\
\mathbb{C}^n & & \mathbb{C}P^{n-1}
\end{tikzcd}$$

$$\omega_\lambda := \pi^* \omega_{std} + \lambda^2 \underline{pr^* \omega_{FS}}$$

$$v \in T Bl_0 \mathbb{C}^n$$

$$\omega_\lambda(v, Jv) = \underline{\omega_{std}(\pi_* v, \pi_*(Jv))} + \lambda^2 pr^* \omega_{FS}(v, Jv) \geq 0$$

$v \neq 0$

1st term is zero $\Longleftrightarrow$ $v \in \underline{\ker d\pi}$, $v \neq 0$

$$\Updownarrow$$

2nd term is non-zero $\Longleftarrow$ $\cancel{\Rightarrow}$ $v \in TE = T\mathbb{C}P^{n-1}$

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2 + \lambda^2)^{1/2}$$

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2 + \lambda^2)^{1/2}$$

is a symplectomorphism.

$\omega_\lambda \in \Omega^2(Bl_0 \mathbb{C}^n)$

ω_λ restricts to a form on $\mathbb{C}^n \backslash \{0\}$,

($\because$ $\pi : Bl_0\mathbb{C}^n \to \mathbb{C}^n$ is a biholo away from $\pi^{-1}(0)$)

which is also called ω_λ

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2 + \lambda^2)^{1/2}$$

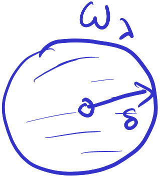
is a symplectomorphism.

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2 + \lambda^2)^{1/2}$$

  is a symplectomorphism.
- Proof :



Towards a symplectic blow-up

Next Class: 10th Nov.

- Lemma : The map $F : (B_\delta \setminus \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \setminus \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2+\lambda^2)^{1/2} =: Z$$

is a symplectomorphism.

- Proof : $z_1 \dots z_n$ on $\mathbb{C}^n$

$$\omega_\lambda = \frac{i}{2}\left(\sum \frac{dz_i \wedge d\bar{z}_i}{\|z\|^2} - \frac{(\sum \bar{z}_i\, dz_i) \wedge (\sum_i z_i\, d\bar{z}_i)}{|z|^4} \right.$$

$$\parallel$$

$$F^*\left(\frac{i}{2} dZ \wedge d\bar{Z}\right)$$

Plug in exp for F

Towards a symplectic blow-up

- Lemma : The map $F : (B_\delta \backslash \{0\}, \omega_\lambda) \to (B_{\sqrt{\delta^2+\lambda^2}} \backslash \overline{B}_\lambda, \omega_{std})$ defined as

$$z \mapsto \frac{z}{\|z\|}(\|z\|^2 + \lambda^2)^{1/2}$$

  is a symplectomorphism.

- Proof :

$$\frac{i}{2}\partial\bar{\partial}\log(|z_1|^2 + \dots + |z_n|^2)$$

$$\frac{i}{2}\partial\left(\frac{\sum_i z_i d\bar{z}_i}{\|z\|^2}\right) = \frac{i}{2}\,\frac{\|z\|^2 \sum_i dz_i \wedge d\bar{z}_i - (z\cdot d\bar{z})(\bar{z}\cdot dz}{\|z\|^4}$$

$$z = (z_1, \dots, z_n)$$