

Introduction to Symplectic Geometry : Lecture 1

August 16, 2021

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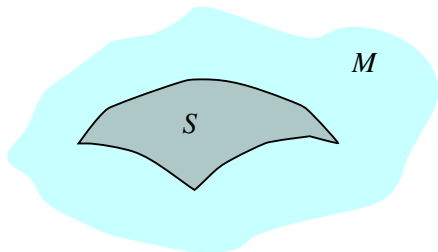
is a bilinear skew-symmetric non-degenerate pairing.

- ▶ $\wedge^n \omega$ is a non-vanishing $2n$ -form.
- Symplectic manifolds are even-dimensional.

Symplectic form

A two-form ω is an area function on surfaces in M : given a surface $S \subset M$, we get an ω -area

$$\omega(S) := \int_S \omega.$$

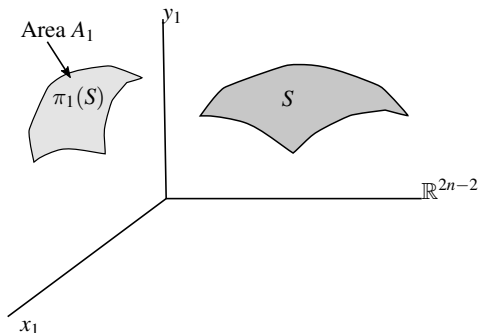


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- The ω -area is $\omega(S) := \sum_{i=1}^n A_i$, where A_i is the area of the projection of S to the (x_i, y_i) -planes.

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- Torus : $\mathbb{T}^{2n} := (\mathbb{R}^{2n}, \omega_0) / \mathbb{Z}^{2n}$,
- Two-sphere : \mathbb{S}^2 with an area form.
- Other examples that we will see : cotangent bundles, non-singular projective varieties, and many others.

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Hamilton's equations

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n.$$

Here $H(p, q) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is the Hamiltonian function.

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- The Hamiltonian is typically the total energy of the system.

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- The quantity $\sum_i dq_i \wedge dp_i$ is conserved. This is the *symplectic form* on \mathbb{R}^{2n} .
- Thus the phase space is a symplectic manifold, and motion is the flow of a vector field that preserves the symplectic form.

Properties of symplectic manifolds : Darboux's theorem

The neighbourhood of any point in a symplectic manifold looks exactly like \mathbb{R}^{2n} with standard symplectic form.

Theorem

Let (M, ω) be a $2n$ -dimensional symplectic manifold. For any point p there is a neighborhood U and coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n) : U \rightarrow \mathbb{R}^{2n}, \quad p \mapsto 0$$

such that $\omega|_U = \sum_i dx_i \wedge dy_i$.

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- Contrast this with Riemannian manifolds.
- A Riemannian manifold comes with an inner product on the tangent space at any point. Thus, on a Riemannian manifold, one can measure length, angle, surface area, volume.
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- Riemannian geometry is more rigid than symplectic geometry : to every point, one can associate a 'curvature'.
- We may say symplectic manifolds are more *flexible* (lacking structure) compared to Riemannian manifolds.

The two-dimensional case

The 2-balls are embedded in Euclidean 3-space. Both inherit an area form and a Riemannian metric from the ambient Euclidean space.



They are the ‘same’ as symplectic manifolds because they have the same area. But they are not the same Riemannian manifolds.

Symplectomorphisms aka Canonical transformations

- A smooth invertible map $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a *symplectomorphism* if it preserves symplectic area.
- That is, for any surface S in \mathbb{R}^{2n} , $\omega(S) = \omega(\phi(S))$.

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- Example :

$$(q_1, p_1, \dots, q_n, p_n) \mapsto (2q_1, \frac{1}{2}p_1, q_2, p_2, \dots, q_n, p_n).$$

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- Another example :

$$(q_1, p_1, \dots, q_n, p_n) \mapsto (q_2, p_2, q_1, p_1, \dots, q_n, p_n).$$

What are some symplectic invariants?

- Volume is an invariant of a symplectic manifold. The symplectic form gives a volume form

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 - ▶ Remark : Not all even-dimensional manifolds have a symplectic form. Example: S^{2n} has a symplectic form only if $n = 1$.

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- The rigidity question: Is symplectic structure much stronger than a volume form?

Another version of the rigidity question

Can any volume preserving map be C^0 -approximated by symplectomorphisms?

Gromov's non-squeezing theorem (1985)

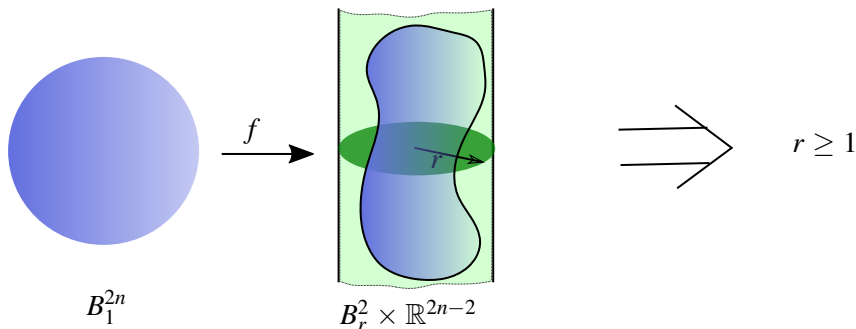
Gromov's non-squeezing theorem

A unit ball in the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ cannot be mapped by a symplectomorphism into any cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ which is narrower than the ball (i.e. $r < 1$).

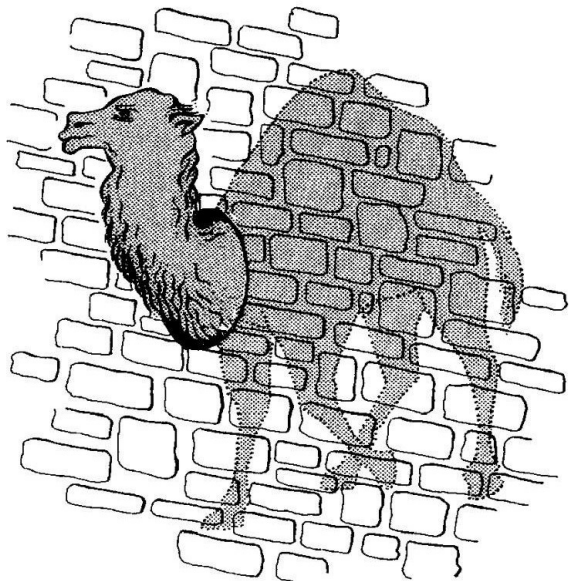
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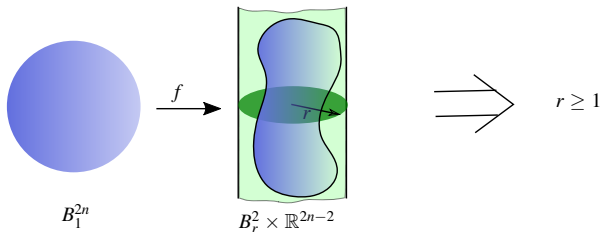
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Can a symplectic camel pass through the eye of a needle?

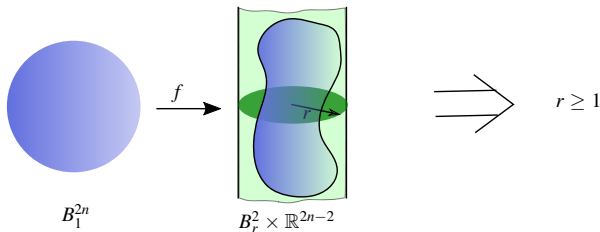


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Therefore, symplectomorphisms can not be approximated by volume preserving diffeomorphisms.

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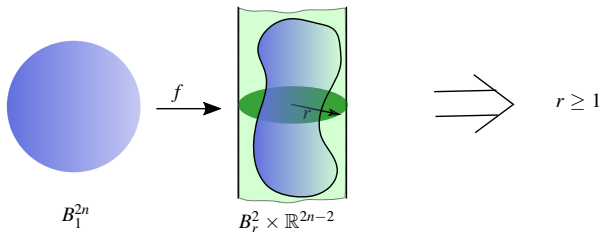


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$$(p_1, q_1, p_2, q_2) \xrightarrow{\phi} (rp_1, rq_1, p_2/r, q_2/r)$$

is volume-preserving and satisfies $\phi(B_1) \subset B_r^2 \times \mathbb{R}^2$.

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is volume-preserving and satisfies $\phi(B_1) \subset B_r^2 \times \mathbb{R}^2$. If $r < 1$, ϕ can not be the limit of symplectomorphisms.

Complex structure on a vector space

- On a real vector space V , a complex structure is a ‘rule’ for multiplication by i . It is a linear map

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- If the vector space V has a symplectic form ω , J is *compatible* with ω if
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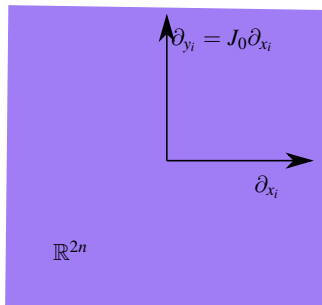
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- The pairing

$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}, \quad (v, w) \mapsto \omega(v, Jw)$$

is a positive definite inner product.

Complex structure on a vector space

The standard complex structure on \mathbb{R}^{2n} is compatible with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$.



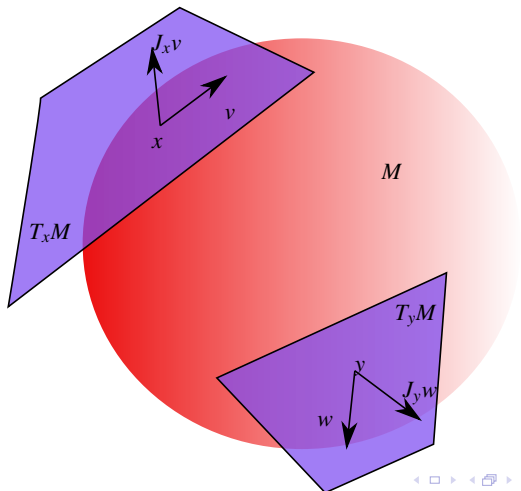
$$J_0 \partial_{x_i} = \partial_{y_i}$$

$$J_0 \partial_{y_i} = -\partial_{x_i}$$

$$\omega_0(\partial_{x_i}, J_0 \partial_{x_i}) = \omega_0(\partial_{x_i}, \partial_{y_i}) > 0$$

Almost complex structure on a manifold

On a symplectic manifold (M, ω) , an almost complex structure consists of a compatible complex structure $J_x : T_x M \rightarrow T_x M$ on every tangent space $(T_x M, \omega_x)$.



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- But there are many choices for a compatible almost complex structure.
- The space of ω -compatible almost complex structures $\mathcal{J}(M, \omega)$ is connected : so any two elements J_0 and J_1 can be connected by a path $\{J_t\}_{t \in [0,1]}$ in $\mathcal{J}(M, \omega)$.

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- Connectedness of $\mathcal{J}(M, \omega)$ implies that certain properties of the rigid space (M, ω, J_0) are preserved in (M, ω, J_1) .
- In fact we will show that $\mathcal{J}(M, \omega)$ is contractible.

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- Since a symplectic manifold can always be equipped with an almost complex structure, some features of complex manifolds carry over to symplectic manifolds.
- In general an almost complex structure is not a complex structure because it may not be ‘integrable’. That is, the manifold (M, J) may not have holomorphic local coordinates.
- This is a Frobenius-like phenomenon.

Digression : Frobenius theorem

- Definition : Let M be a manifold. A sub-bundle $E \subset TM$ is *integrable* if for any point $m \in M$ there is a submanifold $N \subset M$ containing m such that for all $n \in N$ $T_n N = E_n$.

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- Frobenius theorem : E is integrable if and only if $[E, E] \subseteq E$.

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- But not all symplectic manifolds possess an integrable J . We will see an example.

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- Neighborhood theorems such as Darboux theorem : These are local properties of symplectic manifolds.
- Group actions on symplectic manifolds, and in particular torus actions. Symplectic manifolds with certain nice torus actions are actually ‘toric varieties’ and can be classified using certain combinatorial objects.
- Other constructions of symplectic manifolds : Blow-ups, symplectic cut and sum, symplectic fibrations, etc.

J -holomorphic curves

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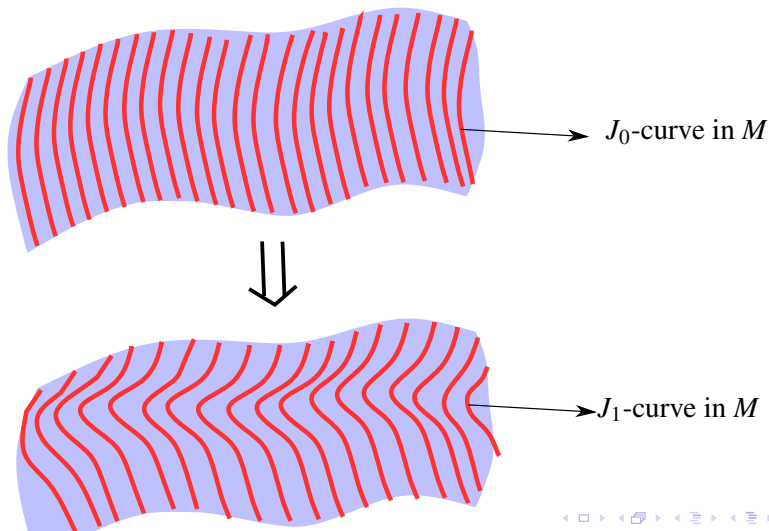
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Result

Suppose J_0, J_1 are compatible almost complex structures on a compact symplectic manifold (M, ω) . For a fixed homology class $\beta \in H_2(M)$ and a generic point p , the number of isolated J_0 -spheres of class β passing through p is same as the number of isolated J_1 -spheres of class β passing through p . (Both counts are signed counts.)

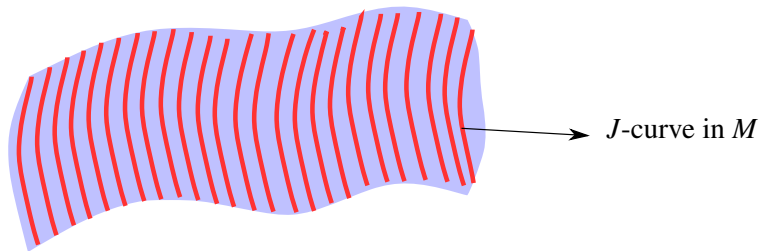
J -holomorphic curves

For example if M is an S^2 -fibration where the fibers are J_0 -holomorphic spheres, then the count is 1 :



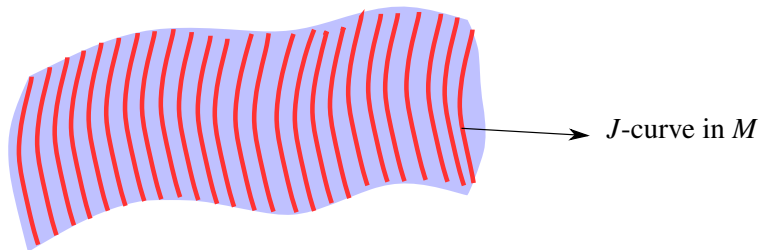
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In general Gromov-Witten invariants are counts of curves of a prescribed homology class satisfying certain constraints, such as passing through some fixed points or cycles.

Gromov-Witten invariant

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Kontsevich produced a recursive formula for the number of degree d curves in \mathbb{P}^2 passing through $3d - 1$ fixed generic points.

Back to Gromov's non-squeezing theorem

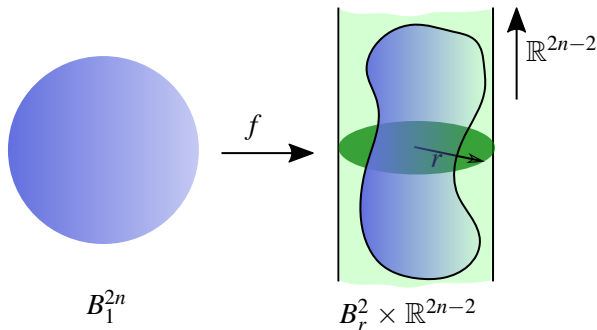
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Proof of the non-squeezing theorem

Assume the contrapositive. Suppose for some $r < 1$ there is a symplectic embedding

$$f : B_1^{2n} \rightarrow B_r^2 \times \mathbb{R}^{2n-2}.$$

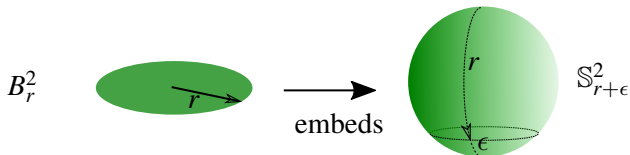


Proof of the non-squeezing theorem

We change the target space to a compact symplectic manifold. Firstly there is a symplectic embedding

$$B_r^2 \rightarrow \mathbb{S}_{r+\epsilon}^2,$$

where $\mathbb{S}_{r+\epsilon}^2$ has area $\pi(r+\epsilon)^2$, and $r < r+\epsilon < 1$.

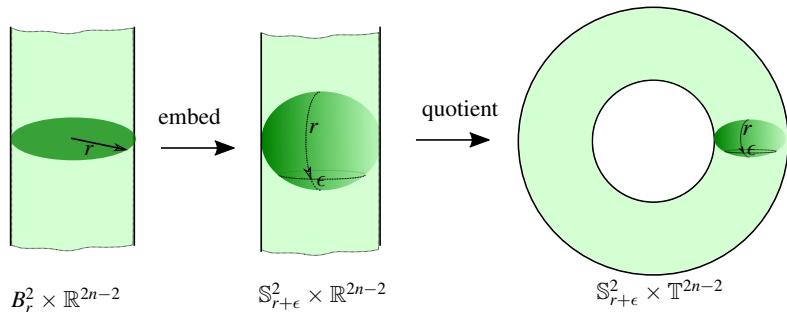


Proof of the non-squeezing theorem

Quotient the factor \mathbb{R}^{2n-2} by a lattice $a\mathbb{Z}^{n-2}$:

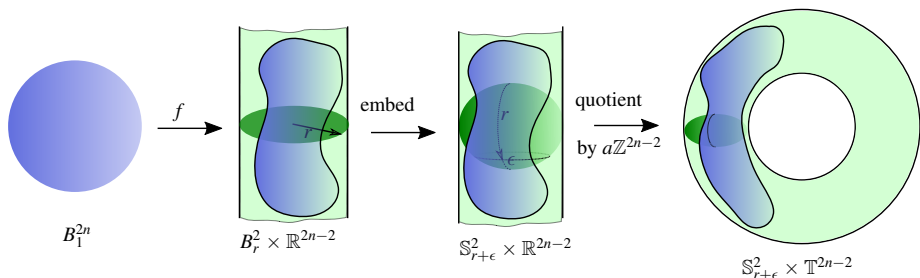
$$\mathbb{R}^{2n-2} / a\mathbb{Z}^{2n-2} \simeq \mathbb{T}^{2n-2},$$

where a is chosen to be large.



Proof of the non-squeezing theorem

The constant a is chosen so that the composed map is an embedding.



Proof of the non-squeezing theorem

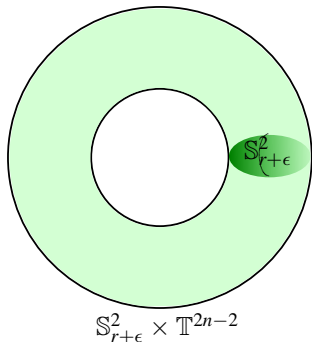
For the product almost complex structure

$$J_{std} := J_{\mathbb{S}^2} \times J_{\mathbb{T}^2}$$

the manifold $M := \mathbb{S}^2 \times \mathbb{T}^{2n-2}$ has a fibration

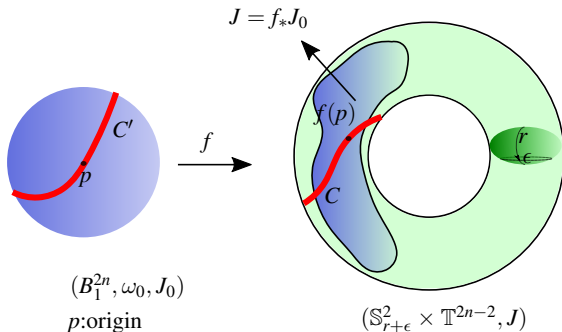
$$(\mathbb{S}^2, J_{\mathbb{S}^2}) \hookrightarrow M \rightarrow \mathbb{T}^{2n-2}$$

whose fibers are J_{std} -holomorphic.
Each of the fibers is \mathbb{S}^2 .



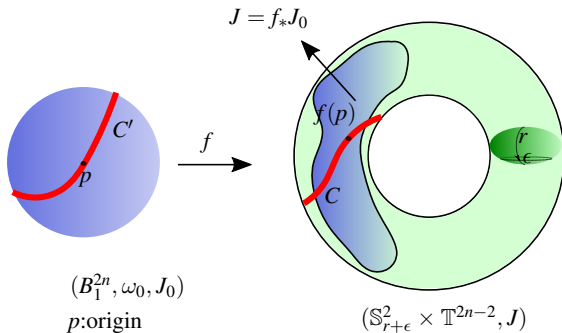
Proof of the non-squeezing theorem

By our earlier result, for any compatible almost complex structure J , there is a J -curve C of homology class $[\mathbb{S}^2] \times pt$ through the point $f(p)$.



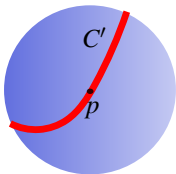
Proof of the non-squeezing theorem

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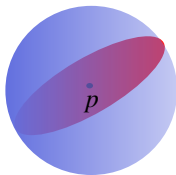
Let $C' := f^{-1}(C)$.

Proof of the non-squeezing theorem



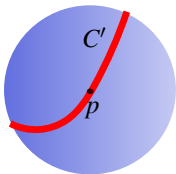
$(B_1^{2n}, \omega_0, J_0)$

p :origin

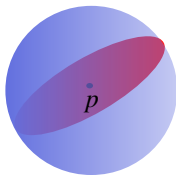


- Holomorphic curves are area minimizers for the metric $\omega(\cdot, J\cdot)$. The metric $\omega_0(\cdot, J_0\cdot)$ is the standard Euclidean metric. Thus C' is a minimal surface in the Euclidean ball.

Proof of the non-squeezing theorem

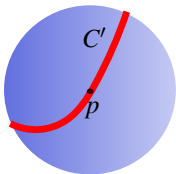


$(B_1^{2n}, \omega_0, J_0)$
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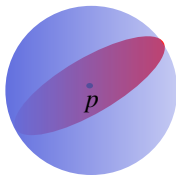


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- By the theory of minimal surfaces, the least area surface in B_1^{2n} with boundary in ∂B_1^{2n} and containing the origin is the flat plane of area π .

Proof of the non-squeezing theorem



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- By the theory of minimal surfaces, the least area surface in B_1^{2n} with boundary in ∂B_1^{2n} and containing the origin is the flat plane of area π .
- Therefore,
$$\pi \leq \text{Area}(C') < \pi(r + \epsilon)^2$$
which contradicts $r + \epsilon < 1$.