Introduction to Symplectic Geometry : Lecture 1

August 16, 2021

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  - on the tangent space T_mM at any point $m \in M$

$$\omega_m : T_mM \times T_mM \to \mathbb{R}$$

    is a bilinear skew-symmetric non-degenerate pairing.
  - $\wedge^n \omega$ is a non-vanishing $2n$-form.

$\hookrightarrow \dim M = 2n$

$\forall v \in T_mM$, $v \neq 0$, $\exists w \in T_mM$, $\omega(v, w) \neq 0$

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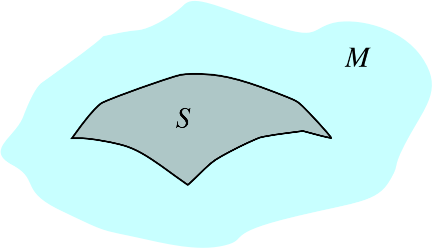
- Symplectic manifolds are even-dimensional.

Symplectic form

A two-form ω is an area function on surfaces in M : given a surface $S \subset M$, we get an ω-area

$$\omega(S) := \int_S \omega.$$



The prototypical example : $\mathbb{R}^{2n}$

$(x_1, y_1, x_2, y_2 \ldots x_n, y_n)$

- The standard symplectic form is $\omega := \sum_{i=1}^{n} dx_i \wedge dy_i$.

$dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots$

$\longleftrightarrow$ n terms

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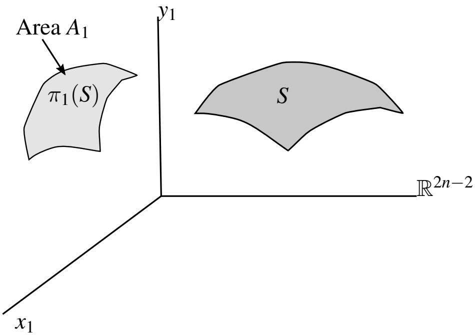

- The ω-area is $\omega(S) := \sum_{i=1}^{n} A_i$, where A_i is the area of the projection of S to the (x_i, y_i)-planes.

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$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

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- Two-sphere : $\mathbb{S}^2$ with an area form.
- Other examples that we will see : cotangent bundles, non-singular projective varieties, and many others.

Motivation : Hamiltonian Mechanics

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Hamilton's equations

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n.$$

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- The Hamiltonian is typically the total energy of the system.

Motivation : Hamiltonian Mechanics

$$H(p,q) = \frac{p^2}{2m} + V(q)$$

K.E. P.E.

Hamilton's equations

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$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m} \qquad \frac{dq_i}{dt} = \frac{p_i}{m}$$

$$\frac{dp}{dt} = \text{Force} = -\text{grad}\, V(q) \Rightarrow \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Motivation : Hamiltonian Mechanics

$X_H := \sum_i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \in Vect(\mathbb{R}^{2n})$

Hamilton's equations

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$x(t) = (p(t), q(t))$.

$\omega = \sum dq_i \wedge dp_i$

$\frac{dx(t)}{dt} = X_H(x(t))$

$i_{X_H}\omega = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i = dH$

Motivation : Hamiltonian Mechanics

$$\iota_{X_H}\omega = \omega(X_H, \cdot) \in \Omega^1(M)$$

Hamilton's equations

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n,$$

Claim

- The quantity $\overbrace{\sum_i dq_i \wedge dp_i}^{\omega}$ is conserved. This is the *symplectic form* on $\mathbb{R}^{2n}$.

$$L_{X_H}\omega = 0$$

$$i_{X_H}\omega = dH \quad \text{Hamilton equation}$$

$(d\omega = 0)$

Proof

$$L_{X_H} = i_{X_H} d + d\, i_{X_H}$$

Cartan's formula

$$L_{X_H}\omega = d\, i_{X_H}\omega = d(dH) = 0 \quad \blacksquare$$

Motivation : Hamiltonian Mechanics

Hamilton's equations

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- The quantity $\sum_i dq_i \wedge dp_i$ is conserved. This is the *symplectic form* on $\mathbb{R}^{2n}$.
- Thus the phase space is a symplectic manifold, and motion is the flow of a vector field that preserves the symplectic form.

Properties of symplectic manifolds : Darboux's theorem

The neighbourhood of any point in a symplectic manifold looks exactly like $\mathbb{R}^{2n}$ with standard symplectic form.

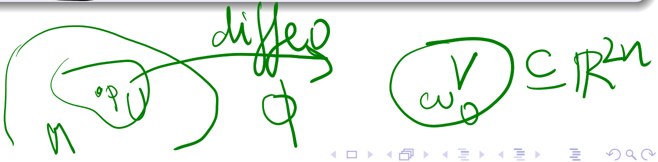
Theorem

Let (M,ω) be a 2n-dimensional symplectic manifold. For any point p there is a neighborhood U and coordinates

$$(x_1,\dots,x_n,y_1,\dots,y_n) : U \to \mathbb{R}^{2n}, \quad p \mapsto 0$$

such that $\omega|U = \sum_i dx_i \wedge dy_i$.

$\omega|_U = \phi^*\omega_0$



Lack of structure

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- Contrast this with Riemannian manifolds.
- A Riemannian manifold comes with an inner product on the tangent space at any point. Thus, on a Riemannian manifold, one can measure length, angle, surface area, volume.
- Riemannian geometry is more rigid than symplectic geometry : to every point, one can associate a 'curvature'.

more structure

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- Riemannian geometry is more rigid than symplectic geometry : to every point, one can associate a 'curvature'.
- We may say symplectic manifolds are more *flexible* (lacking structure) compared to Riemannian manifolds.

The two-dimensional case

The 2-balls are embedded in Euclidean 3-space. Both inherit an area form and a Riemannian metric from the ambient Euclidean space.

They are the ‘same’ as symplectic manifolds because they have the same area. But they are not the same Riemannian manifolds.

Symplectomorphisms aka Canonical transformations

- A smooth invertible map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a *symplectomorphism* if it preserves symplectic area.
- That is, for any surface S in $\mathbb{R}^{2n}$, $\omega(S) = \omega(\phi(S))$.

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- Example :

$$(q_1, p_1, \dots, q_n, p_n) \overset{\phi}{\mapsto} (2q_1, \frac{1}{2}p_1, q_2, p_2, \dots, q_n, p_n).$$

$$dq_1 \wedge dp_1 + \dots$$

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diffeomorphism

$\phi^* \omega = \omega$

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- Another example :

$$(q_1, p_1, \ldots, q_n, p_n) \mapsto (q_2, p_2, q_1, p_1, \ldots, q_n, p_n).$$

What are some symplectic invariants?

$\dim(M) = 2n$

- Volume is an invariant of a symplectic manifold. The symplectic form gives a volume form

$$\text{vol} := \omega^n := \omega \wedge \cdots \wedge \omega. \in \Omega^{2n}(M)$$

M: compact

$$\text{Volume}(M) := \int_M \text{vol}$$

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$x, y \mapsto 0, x, y, 0$

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- The deRham cohomology class $[\omega] \in H^2(M)$ is an invariant.

$d\omega = 0$

Ex: $\mathbb{R}^4$, $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$

Find a 2-d submfd that is not symp

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- The deRham cohomology class $[\omega] \in H^2(M)$ is an invariant.
  - Remark : Not all even-dimensional manifolds have a symplectic form. Example: S^{2n} has a symplectic form only if $n = 1$.

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- There are examples of symplectic manifolds (M, ω_0), (M, ω_1) that are not symplectomorphic and $[\omega_0] = [\omega_1] \in H^2(M)$.
- The rigidity question: Is symplectic structure much stronger than a volume form?

Another version of the rigidity question

Can any volume preserving map be C^0-approximated by symplectomorphisms?

Gromov's non-squeezing theorem (1985)

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A unit ball in the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ cannot be mapped by a symplectomorphism into any cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ which is narrower than the ball (i.e. $r < 1$).

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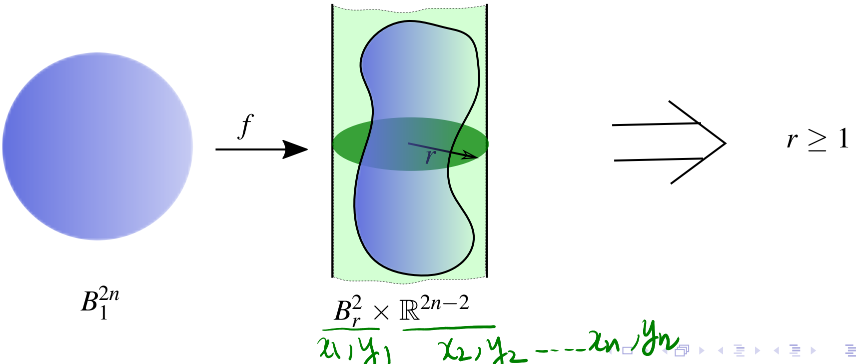
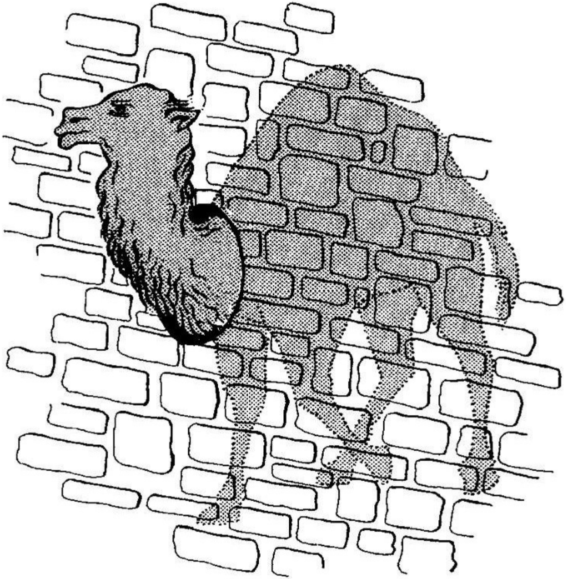
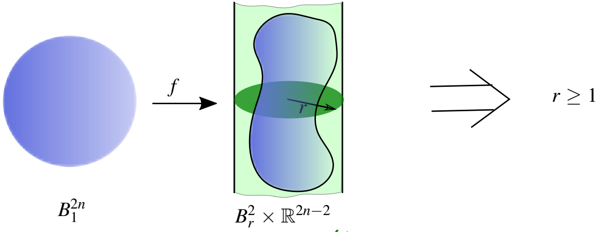


Can a symplectic camel pass through the eye of a needle?



Gromov's non-squeezing theorem



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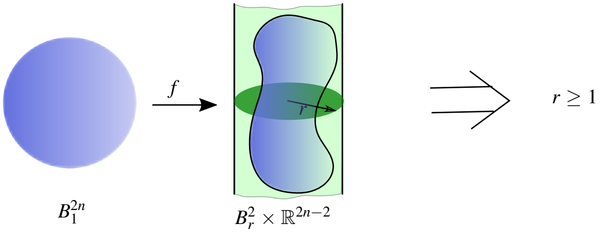

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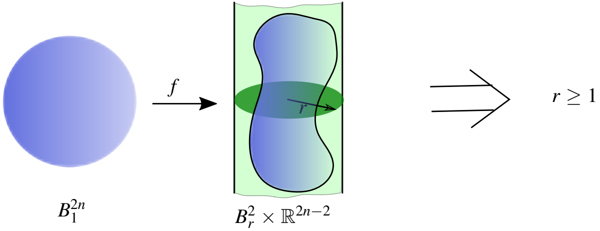
$$\mathbb{R}^4 \quad (p_1, q_1, p_2, q_2) \stackrel{\phi}{\mapsto} (rp_1, rq_1, p_2/r, q_2/r) \quad \mathbb{R}^4$$

is volume-preserving and satisfies $\phi(B_1) \subset B_r^2 \times \mathbb{R}^2$.

Gromov's non-squeezing theorem



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is volume-preserving and satisfies $\phi(B_1) \subset B_r^2 \times \mathbb{R}^2$. If $r < 1$, ϕ can not be the limit of symplectomorphisms.