

Introduction to Symplectic Geometry : Lecture 18

October 18, 2021

Splittings of the complexified cotangent bundle

- Let (M, J) be an almost complex manifold. Consider the complexified bundle $T_{\mathbb{C}}^*M = T^*M \otimes \mathbb{C}$. The space of sections

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- In a similar way, $T_{\mathbb{C}}^*M = T^{1,0}M \oplus T^{0,1}M$. The summands $T^{1,0}M, T^{0,1}M$ are complex vector bundles.

Splittings of the complexified cotangent bundle

- The space of sections

$$\Omega^{1,0}(M), \quad \text{resp.} \quad \Omega^{0,1}(M)$$

is the space of $(1, 0)$ -forms resp. $(0, 1)$ -forms.

- On a chart U in a complex manifold X with holomorphic charts z_1, \dots, z_n ,

$$\Omega^{1,0}(M) = \left\{ \sum_i f_i(z) dz_i : f_i : U \rightarrow \mathbb{C} \text{ is a smooth map} \right\},$$

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- Here $dz_i = dx_i + idy_i$, $d\bar{z}_i = dx_i - idy_i$.

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Splittings of the space of k -forms

- Let (M, J) be an almost complex manifold. Since $\Omega^1(M, \mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}$, we have

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Define $\Omega^{l,m}(M) := (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1})$.

- On a chart (U, \underline{z}) of a complex manifold,

$$\Omega^{l,m}(M) = C^\infty(U, \mathbb{C})\text{-span of } \{dz_L \wedge d\bar{z}_M : |L| = l, |M| = m\}$$

Splitting of the exterior derivative

- On $C^\infty(M, \mathbb{C})$, $d = \partial + \bar{\partial}$, where

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- On a complex manifold, $\partial^2 = 0$, $\bar{\partial}^2 = 0$, $\bar{\partial}\partial + \partial\bar{\partial} = 0$.

Kähler manifold

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- Result : A manifold has an almost complex structure iff it has a non-degenerate 2-form.

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So, $-2i\omega(v_{1,0}, v_{0,1}) = (v + iw)^* h(v + iw)$. □

Thus for any point z , $(h_{jk}(z))_{j,k}$ is a positive definite Hermitian matrix.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \rightarrow \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial z_k})_{j,k}$ is positive definite.

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- Example : $f = \|z\|^2$ on \mathbb{C}^n gives the form

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