Introduction to Symplectic Geometry : Lecture 18

October 18, 2021

M : $\mathbb{C}$-mfd $\qquad N \subseteq M$ real submfd

$T_n N \subseteq T_n M$ is a complex subspace $\forall n \in N$

Q : N is complex ?

Splittings of the complexified cotangent bundle

- Let (M, J) be an almost complex manifold. Consider the complexified bundle $T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$. The space of sections

$$\Gamma(M, T^*_{\mathbb{C}}M) = \Omega^1(M, \mathbb{C})$$

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- In a similar way, $T^*_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. The summands $T^{1,0}M$, $T^{0,1}M$ are complex vector bundles.

Splittings of the complexified cotangent bundle

- The space of sections

$$\Omega^{1,0}(M), \quad \text{resp.} \quad \Omega^{0,1}(M)$$

is the space of $(1,0)$-forms resp. $(0,1)$-forms.

- On a chart U in a complex manifold X with holomorphic charts $z_1, \ldots, z_n$,

$$\Omega^{1,0}(M) = \{\sum_i f_i(z) dz_i : f_i : U \to \mathbb{C} \text{ is a smooth map}\},$$

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- Here $dz_i = dx_i + i dy_i$, $d\bar{z}_i = dx_i - i dy_i$.

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$$T_{1,0}U = C^{\infty}(U, \mathbb{C})\text{-span of } \{\frac{\partial}{\partial z_i}\}_i,$$

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Splittings of the space of k-forms

- Let (M, J) be an almost complex manifold. Since $\Omega^1(M, \mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}$, we have

$$\Omega^k(M, \mathbb{C}) = \sum_{l+m=k} (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1}).$$

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  Define $\Omega^{l,m}(M) := (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1})$.

- On a chart $(U, \underline{z})$ of a complex manifold,

$$\Omega^{l,m}(M) = C^\infty(U,\mathbb{C})\text{-span of } \{dz_L \wedge d\bar{z}_M : |L| = l, |M| = m\}$$

Splitting of the exterior derivative

- On $C^\infty(M,\mathbb{C})$, $d = \partial + \overline{\partial}$, where

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$$\partial f = \sum_i \frac{\partial f}{\partial z_i} dz_i, \quad \overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i.$$

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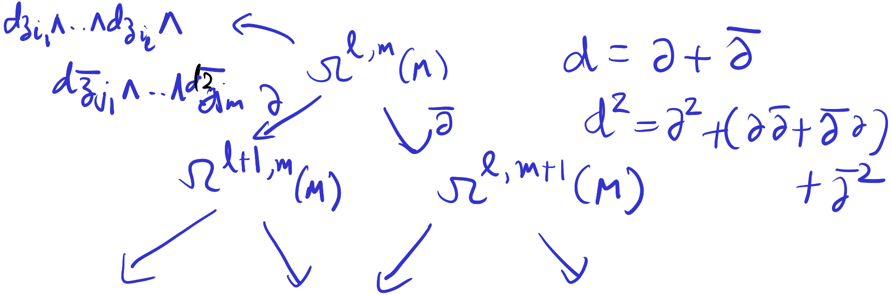
and thus, $d = \partial + \overline{\partial}$ for all forms on a complex manifold.
- On a complex manifold, $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\overline{\partial}\partial + \partial\overline{\partial} = 0$.



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- (Symplectic viewpoint) : A Kähler manifold is a symplectic manifold (M, ω) and an ω-compatible almost complex structure such that (M, J) is a complex manifold.

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- A Hermitian metric H on a complex manifold is point-wise a positive definite sesquilinear product

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- Result : A manifold has an almost complex structure iff it has a non-degenerate 2-form.

From complex to Kähler

On a complex manifold (M, J), $\omega \in \Omega^2(M, \mathbb{C})$ is a Kähler form if the following are satisfied:

- ω is J-invariant

  $\omega(Ju, Jv) = \omega(u, v)$

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NO $\Omega^{0,2}$, $\Omega^{2,0}$ - parts

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- $d\omega = 0$

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2,1 1,2

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- ω is non-degenerate $\Leftrightarrow \det(h_{jk}(z))_{j,k} \neq 0$. Ex
- J is ω-tame

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n} = \left(\frac{i}{2}\right)^n \det(h_{jk})\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

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- J is ω-tame $\Leftrightarrow$ (h_{jk}) is positive definite

on TM

$\forall v \in \mathbb{C}^n \setminus \{0\}$

$v^*(h_{jk}) v > 0$

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- Let $v = \sum_j v_j\partial_{x_j} + w_j\partial_{y_j}$. Then

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- Let $v = \sum_j v_j \partial_{x_j} + w_j \partial_{y_j}$. Then

$$v_{1,0} = \sum_j v_j \partial_{z_j} + i w_j \partial_{\bar{z}_j}, \quad v_{0,1} = \sum_j v_j \partial_{\bar{z}_j} - i w_j \partial_{\bar{z}_j}.$$

So, $-2i\omega(v_{1,0}, v_{0,1}) = (v + iw)^* h(v + iw)$. □

Thus for any point z, $(h_{jk}(z))_{j,k}$ is a positive definite Hermitian matrix.

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $(\frac{\partial^2 f}{\partial z_j \partial \overline{z_k}})_{j,k}$ is positive definite.

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- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$,

$$\omega = \frac{i}{2} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

$$\Omega^0 \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\partial} \Omega^{1,1}$$

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- Claim : Then $\omega := \frac{i}{2}\partial\bar{\partial} f$ is a Kähler form, and f is called a local Kähler potential.
- Proof of Claim : Check $\omega \in \Omega^{1,1}(U)$, $\omega \in \Omega^2(U, \mathbb{R})$,

$$\bar{\omega} = \frac{-i}{2}\bar{\partial}\partial f = \frac{i}{2}\partial\bar{\partial} f = \omega$$

$$\partial\bar{\partial} + \bar{\partial}\partial = 0$$

Local Kähler potential

- Let $U \subset \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{R}$ is pluri-subharmonic if $\left(\frac{\partial^2 f}{\partial z_j \partial z_k}\right)_{j,k}$ is positive definite.
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$h_{jk} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}$

$\omega = \frac{i}{2} h_{jk}\, dz_j \wedge d\bar{z}_k$
$\Rightarrow (h_{jk})$ is Hermitian matrix

$\partial\omega = \partial\left(\frac{i}{2}\partial\bar{\partial}f\right) = \frac{i}{2}\partial^2\bar{\partial}f = 0$

$\bar{\partial}\omega = \bar{\partial}\left(-\frac{i}{2}\bar{\partial}\partial f\right) = -\frac{i}{2}\bar{\partial}^2\partial f = 0$

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- Example : $f = \|z\|^2$ on $\mathbb{C}^n$ gives the form

$$\frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}$$

$$f(z) = \sum_{j=1}^{n} z_j \overline{z_j}$$

Verify

$$\overline{\partial} z_j = 0 \qquad \partial z_j = dz_j$$

$$\overline{\partial}\, \overline{z_j} = d\overline{z_j} \qquad \partial \overline{z_j} = 0$$

$$\overline{\partial} f = \sum_{j=1}^{n} z_j \, d\overline{z_j}$$

$$\partial\overline{\partial} f = \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$$

Local Kähler potential

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$$\omega = \frac{i}{2}\partial\bar{\partial} f = \frac{i}{2}\sum dz_j \wedge d\bar{z}_j$$

$(h_{jk})_{j,k} = Id$ is positive def and Hermitian

$(\mathbb{C}^n, \omega, J_{std})$ is a Kähler mfd

Local Kähler potential

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- Example $\mathbb{P}^n$: On a chart $[z_0, \dots, z_i = 1, \dots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check). → HW

Define $U_i \subseteq \mathbb{P}^n$ as $U_i := \{z_i \neq 0\}$

skip z_i

$U_i \to \mathbb{C}^n$ Affine chart $\quad [z_0 : \dots : z_n] \mapsto \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots, \frac{z_n}{z_i}\right) \in \mathbb{C}^n$

Local Kähler potential

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Local Kähler potential

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- The forms on all the charts patch:
- Identify the chart $U_0 := \{z_0 \neq 0\}$ to $\mathbb{C}^n$ by

$$[1 : z_1 : \cdots : z_n] \mapsto (z_1, \ldots, z_n)$$

do the same for any U_i.

Local Kähler potential

- Example $\mathbb{P}^n$: On a chart $[z_0, \dots, z_i = 1, \dots, z_n]$ take $f = \log(1 + \sum_{j \neq i} |z_j|^2)$ is a local Kähler potential (check).
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$U_0 \simeq \mathbb{C}^n$

$$[1 : z_1 : \cdots : z_n] \mapsto (z_1, \dots, z_n)$$

do the same for any U_i.

$\to z_1 \neq 0$

- The map between the charts on U_0 and U_1 is given by

$$U_0 \xrightarrow{\phi} U_1, \quad (Z_1, \dots, Z_n) \mapsto (z_0 := \frac{1}{Z_1}, z_2 := \frac{Z_2}{Z_1}, \dots, z_n := \frac{Z_n}{Z_1}).$$

$[1 : Z_1 : Z_2 : \dots : Z_n]$ $\quad Z_1 \neq 0$

Local Kähler potential

$\phi^* f(x) dx_1 \wedge \dots \wedge dx_n = f(x \circ \phi) \, d(x_1 \circ \phi) \wedge \dots \wedge d(x_n \circ \phi)$

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- The form on U_1 is $\omega_1 := \frac{i}{2}\partial\bar{\partial}\log(1 + |z_0|^2 + |z_2|^2 + \cdots + |z_n|^2)$.

$$\phi^* \omega_1 = \frac{i}{2} \partial \bar{\partial} \log\left(1 + \frac{1}{|Z_1|^2} + \frac{|Z_2|^2}{|Z_1|^2} + \dots + \frac{|Z_n|^3}{|Z_1|^2}\right)$$

Local Kähler potential

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Form on U_1 → 0

Local Kähler potential

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$\partial\overline{\partial}(\log z_1 + \log \overline{z_1}) = 0$

Local Kähler potential

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