Introduction to Symplectic Geometry : Lecture 18

October 18, 2021

Recall : A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

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  iff the moment equation holds for $\mathrm{Ad}_{g^{-1}}\xi$.
- The result follows because the Ad-orbit of any $\xi \in \mathfrak{u}(n)$ intersects $\mathfrak{t}$.

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- $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}$, and the quotient

$$\{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}/U(k)$$

is the Grassmanian $\mathrm{Gr}(k, n)$ of k-planes in $\mathbb{C}^n$.

Splittings of the complexified cotangent bundle

- Let (M, J) be an almost complex manifold. Consider the complexified bundle $T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$. The space of sections

$$\Gamma(M, T^*_{\mathbb{C}}M) = \Omega^1(M, \mathbb{C})$$

is the space of complex-valued one-forms.

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- In a similar way, $T^*_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. The summands $T^{1,0}M$, $T^{0,1}M$ are complex vector bundles.

Splittings of the complexified cotangent bundle

- The space of sections

$$\Omega^{1,0}(M), \quad \text{resp.} \quad \Omega^{0,1}(M)$$

is the space of $(1,0)$-forms resp. $(0,1)$-forms.

- On a chart U in a complex manifold X with holomorphic charts $z_1, \ldots, z_n$,

$$\Omega^{1,0}(M) = \{\sum_i f_i(z) dz_i : f_i : U \to \mathbb{C} \text{ is a smooth map}\},$$

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Splittings of the space of k-forms

- Let (M, J) be an almost complex manifold. Since $\Omega^1(M, \mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}$, we have

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  Define $\Omega^{l,m}(M) := (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1})$.

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Splitting of the exterior derivative

- On $C^{\infty}(M, \mathbb{C})$, $d = \partial + \overline{\partial}$, where

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and thus, $d = \partial + \overline{\partial}$ for all forms on a complex manifold.
- On a complex manifold, $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\overline{\partial}\partial + \partial\overline{\partial} = 0$.

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- A Hermitian metric H on a complex manifold is point-wise a positive definite sesquilinear product

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- Result : A manifold has an almost complex structure iff it has a non-degenerate 2-form.

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Thus for any point z, $(h_{jk}(z))_{j,k}$ is a positive definite Hermitian matrix.