Introduction to Symplectic Geometry : Lecture 18

October 18, 2021

Recall : A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

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- $\mathrm{Ad}_{U(n)}\,\mathfrak{t} = \mathfrak{u}(n)$.
- For any $\xi \in \mathfrak{u}(n)$, and $g \in U(n)$,
  $d\mu_\xi = -i_{\xi_M}\omega$ iff $L_g^* d\mu_\xi = -L_g^*(i_{\xi_M}\omega)$

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  iff $d\mu_{\mathrm{Ad}_{g^{-1}}\xi} = i_{dL_g^{-1}(\xi_M)}\omega$
  iff the moment equation holds for $\mathrm{Ad}_{g^{-1}}\xi$.
- The result follows because the Ad-orbit of any $\xi \in \mathfrak{u}(n)$ intersects $\mathfrak{t}$.

$\mathrm{Ad}\,\mathfrak{t} = \mathfrak{u}(n)$

$\xi \in \mathfrak{u}(n)$ $\quad \xi^* + \xi = 0$

$$\mathrm{Ad}_{U(n)} \mathfrak{t} = \mathfrak{u}(n)$$

$$\Updownarrow$$

For $\xi \in \mathfrak{u}(n)$, $\exists P \in U(n)$ $P \xi P^{-1}$ is diagonal $\in \mathfrak{t}$.

$$\xi^* + \xi = 0$$

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$$= \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \cdots & \bar{z}_k \end{pmatrix}$$

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- Result : Let $k \leq n$. The standard action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_{std})$ is

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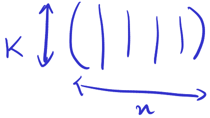

$G \circlearrowright M_1$, $G \circlearrowright M_2$

$$\mathbb{C}^{K\times n} = \mathbb{C}^K \times \mathbb{C}^K \times \dots \times \mathbb{C}^K \quad (n)$$

$$\mu : M_1 \times M_2 \to \mathfrak{g}^\vee$$

$$\mu = \mu_1 + \mu_2$$

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- We may translate by an element in the center of $\mathfrak{u}(k)^\vee$ to get

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- $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}$, and the quotient

$$\{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}/U(k)$$

is the Grassmanian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$.

Splittings of the complexified cotangent bundle

- Let (M, J) be an almost complex manifold. Consider the complexified bundle $T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$. The space of sections

$$\Gamma(M, T^*_{\mathbb{C}}M) = \Omega^1(M, \mathbb{C})$$

is the space of complex-valued one-forms.

$\dim M = 2n$

$\mathrm{RK}(T^*_{\mathbb{C}}M) = 4n$

$\sum_{i=1}^{2n} f_i(x)\, dx_i$

$f_i : U \to \mathbb{C}$

$\eta_i \in T^*_x M$

$\eta \in T^*_x M \otimes \mathbb{C}$

$\eta_1 \otimes 1 + \eta_2 \otimes i$

$\eta_1 \circ J$

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- In a similar way, $T^*_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. The summands $T^{1,0}M, T^{0,1}M$ are complex vector bundles.

$T^*M \simeq T^{1,0}M \simeq T^{0,1}M$

Splittings of the complexified cotangent bundle

- The space of sections $\Gamma(M, T^{1,0}M)$ resp. $\Gamma(M, T^{0,1}M)$

$$\Omega^{1,0}(M), \quad \text{resp.} \quad \Omega^{0,1}(M)$$

  is the space of $(1,0)$-forms resp. $(0,1)$-forms.

- On a chart U in a complex manifold X with holomorphic charts $z_1, \ldots, z_n$,

$$\Omega^{1,0}(M) = \{\sum_i f_i(z) dz_i : f_i : U \to \mathbb{C} \text{ is a smooth map}\},$$

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(X, J) $\quad \dim_{\mathbb{R}} X = 2n$

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- Here $dz_i = dx_i + idy_i$, $d\bar{z}_i = dx_i - idy_i$.

$z_i = x_i + i y_i$
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- On a chart $(U, \underline{z})$ of a complex manifold,

$$T_{1,0}U = C^{\infty}(U, \mathbb{C})\text{-span of } \{\frac{\partial}{\partial z_i}\}_i,$$

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$TM \xrightarrow{\pi_{1,0}} T_{1,0}M$

$v \mapsto \frac{1}{2}(v - Jv \otimes i)$

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$$\pi_{1,0}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right). = \pi_{0,1}\left(\frac{\partial}{\partial x_i}\right)$$

Splittings of the space of k-forms

- Let (M, J) be an almost complex manifold. Since $\Omega^1(M, \mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}$, we have

$$\Omega^k(M, \mathbb{C}) = \sum_{l+m=k} (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1}).$$

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  Define $\Omega^{l,m}(M) := (\wedge^l \Omega^{1,0}) \wedge (\wedge^m \Omega^{0,1})$.

- On a chart $(U, \underline{z})$ of a complex manifold,

$$\Omega^{l,m}(M) = C^\infty(U,\mathbb{C})\text{-span of } \{dz_L \wedge d\bar{z}_M : |L| = l, |M| = m\}$$

$\wedge^k(\Omega^{1,0} \oplus \Omega^{0,1})$

$= \sum_{l+m=k} \Omega^{l,m}(M)$

L, M : mult-indices

$f(z)\, dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2$

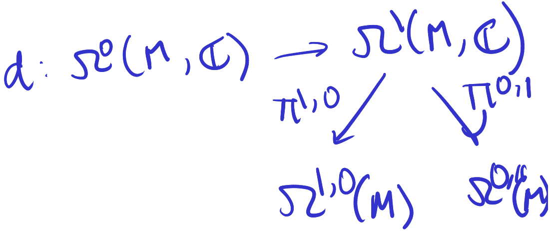
$(2,2)$-form

$f : U \to \mathbb{C}$

Splitting of the exterior derivative

- On $C^\infty(M,\mathbb{C})$, $d = \partial + \overline{\partial}$, where

$$\partial := \pi^{1,0} \circ d, \quad \overline{\partial} = \pi^{0,1} \circ d.$$



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$$\partial := \pi^{1,0} \circ d, \quad \overline{\partial} = \pi^{0,1} \circ d.$$

So $\partial : C^\infty(M, \mathbb{C}) \to \Omega^{1,0}(M)$, $\overline{\partial} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(M)$.

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- On a chart $(U, \bar{z})$ of a complex manifold

$$\partial f = \sum_i \frac{\partial f}{\partial z_i} dz_i, \quad \overline{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

$$\frac{\partial f}{\partial z_i} = \frac{1}{2}\left(\frac{\partial f}{\partial x_i} - \quad \frac{\partial f}{\partial y_i} \right)$$

Splitting of the exterior derivative

- Show that for any $\omega \in \Omega^{l,m}(M)$, $d\omega \in \Omega^{l+1,m}(M) \oplus \Omega^{l,m+1}(M)$.

$$\omega = \sum f(x)\, dz_{i_1} \wedge \dots \wedge dz_{i_l} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_m}$$

$$\Omega^{l,m} \xrightarrow{d} \Omega^{l+1,m} \oplus \Omega^{l,m+1}$$

$$df = \sum \frac{\partial f}{\partial z_i} dz_i + \sum \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

$$d\omega = \sum df \wedge dz_{i_1} \cdots dz_{i_l} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_m}$$

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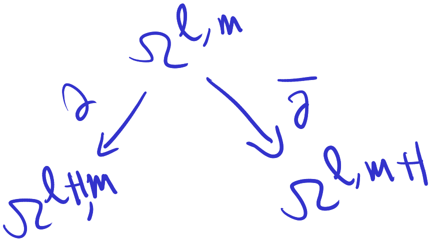
$$\partial\omega := \pi^{l+1,m}(d\omega), \overline{\partial}\omega := \pi^{l,m+1}(d\omega),$$

and thus, $d = \partial + \overline{\partial}$ for all forms on a complex manifold.



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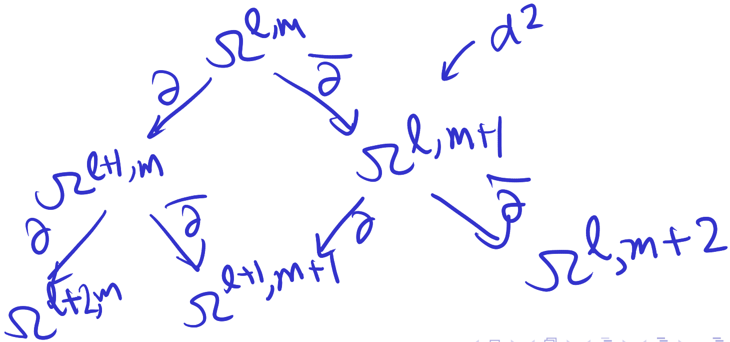
  and thus, $d = \partial + \overline{\partial}$ for all forms on a complex manifold.

- On a complex manifold, $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\overline{\partial}\partial + \partial\overline{\partial} = 0$. $\Leftarrow d^2 = 0$



Kähler manifold

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$$H_x : T_{x,\mathbb{C}}M \times T_{x,\mathbb{C}}M \to \mathbb{C}.$$

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On TM, $h = g + i\omega$ where g is a Riemannian metric and $\omega \in \Omega^2(M, \mathbb{R})$ is a non-degenerate 2-form.

$d\omega = 0$

ω : Kähler form

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  On TM, $h = g + i\omega$ where g is a Riemannian metric and $\omega \in \Omega^2(M, \mathbb{R})$ is a non-degenerate 2-form. (Check : J is ω-compatible.)
- Result : A manifold has an almost complex structure iff it has a non-degenerate 2-form.

Recall - In proofs about $\mathcal{J}(M,\omega)$ we did not use $d\omega = 0$

From complex to Kähler

On a complex manifold (M, J), $\omega \in \Omega^2(M, \mathbb{C})$ is a Kähler form if the following are satisfied:

- ω is J-invariant

  $\omega(Ju, Jv) = \omega(u, v)$

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$$J^*\omega = \omega$$

$$J^* dz_j = i\, dz_j \qquad J^* d\bar{z}_j = -i\, d\bar{z}_j$$

$$\omega \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$$

$$J^*\omega^{2,0} = -\omega^{2,0} \qquad \downarrow \qquad J^*\omega^{0,2} = -\omega^{0,2}$$

$$J^*\omega^{1,1} = \omega^{1,1}$$

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- ω is J-invariant $\Leftrightarrow \omega \in \Omega^{1,1}(M)$. Let $\omega = \frac{i}{2} \sum_{j,k} h_{jk}(z) dz_j \wedge d\bar{z}_k$.

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m TM

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- $\omega \in \Omega^2(M, \mathbb{R}) \Leftrightarrow h_{jk} = \bar{h}_{kj}$.

$(h_{jk})_{j,k}$ is a Hermitian matrix

$\omega = \bar{\omega}$

$= \overline{-\frac{i}{2} \sum_{j,k} h_{jk} d\bar{z}_j \wedge dz_k}$

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$\Omega^{2,1}(M)$ $\longrightarrow$ $\in \Omega^{1,2}(M)$

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- $d\omega = 0 \Leftrightarrow \partial\omega = \bar{\partial}\omega = 0$.
- ω is non-degenerate $\Leftrightarrow \det(h_{jk}(z))_{j,k} \neq 0$.
- J is ω-tame

$\underbrace{\omega \wedge \dots \wedge \omega}_{n} = \left(\frac{i}{2}\right)^n \det(h_{jk})\, dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$

(Check signs)

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- ω is non-degenerate $\Leftrightarrow \det(h_{jk}(z))_{j,k} \neq 0$.
- J is ω-tame $\Leftrightarrow (h_{jk}(z))_{j,k}$ is positive definite.

Thus for any point z, $(h_{jk}(z))_{j,k}$ is a positive definite Hermitian matrix.

$\omega(v, Jv) > 0$

$g := \omega(\cdot, J\cdot)$

$H := g + i\omega$

$v \in TM$

$v = \sum v_i \frac{\partial}{\partial z_i} + \sum w_i \frac{\partial}{\partial \bar{z}_i}$

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$\omega(v, Jv)$

$=$

$v \mapsto \frac{1}{2}(v - Jv \otimes i)$

$TM \to T_{1,0}M$

$\searrow T_{0,1}M$

$v \mapsto \frac{1}{2}(v + Jv \otimes i)$

ANNOUNCEMENT

No Class on

27 Oct 1 Nov 3 Nov
8 Nov

Class on 10 Nov (Wed), 12 Nov (Fri)