Introduction to Symplectic Geometry : Lecture 17

October 13, 2021

Recall : Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

$$i^*\omega = \pi^*\overline{\omega}$$

    where $i : \mu^{-1}(0) \to M$ is the inclusion map.

Recall : Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

$$i^*\omega = \pi^*\overline{\omega}$$

    where $i : \mu^{-1}(0) \to M$ is the inclusion map.
- $\overline{M}$ is called the **symplectic quotient** at level 0.

Recall : Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

$$i^*\omega = \pi^*\overline{\omega}$$

    where $i : \mu^{-1}(0) \to M$ is the inclusion map.
- $\overline{M}$ is called the **symplectic quotient** at level 0.
- Example : Consider the diagonal action

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \ldots, z_n)$$

  with moment map $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2}$.

Recall : Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

    $$i^*\omega = \pi^*\overline{\omega}$$

    where $i : \mu^{-1}(0) \to M$ is the inclusion map.
- $\overline{M}$ is called the **symplectic quotient** at level 0.
- Example : Consider the diagonal action

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \ldots, z_n)$$

with moment map $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2}$. The quotient is $\mathbb{P}^{n-1}$. We have thus showed that the complex projective space has a symplectic form.

Recall : Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

$$i^*\omega = \pi^*\overline{\omega}$$

    where $i : \mu^{-1}(0) \to M$ is the inclusion map.
- $\overline{M}$ is called the **symplectic quotient** at level 0.
- Example : Consider the diagonal action

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \dots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \dots, z_n)$$

  with moment map $\mu(z_1, \dots, z_n) = \frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2}$. The quotient is $\mathbb{P}^{n-1}$. We have thus showed that the complex projective space has a symplectic form.

Motivation for symplectic reduction : Noether's principle

- Let (M, ω, G, μ) be a Hamiltonian G-space.

Motivation for symplectic reduction : Noether's principle

- Let (M, ω, G, μ) be a Hamiltonian G-space.
- Theorem (Noether) : A Hamiltonian function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the Hamiltonian vector field v_H.

Motivation for symplectic reduction : Noether's principle

- Let (M, ω, G, μ) be a Hamiltonian G-space.
- Theorem (Noether) : A Hamiltonian function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the Hamiltonian vector field v_H.
- Proof : For any $\xi \in \mathfrak{g}$,

$$d\mu_\xi(v_H) = -\omega(\xi_M, v_H) = -dH(\xi_M).$$

Motivation for symplectic reduction : Noether's principle

- Let (M, ω, G, μ) be a Hamiltonian G-space.
- Theorem (Noether) : A Hamiltonian function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the Hamiltonian vector field v_H.
- Proof : For any $\xi \in \mathfrak{g}$,

$$d\mu_\xi(v_H) = -\omega(\xi_M, v_H) = -dH(\xi_M).$$

- Noether's theorem implies that one can study the dynamics of v_H on the symplectic quotient $\mu^{-1}(c)/G$.

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee$,

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.
- μ_1 generates the G_1-action and μ_2 generates the G_2-action.

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.
- μ_1 generates the G_1-action and μ_2 generates the G_2-action.
- Claim : μ_2 is constant on a G_1-orbit, and similarly μ_1 is constant on a G_2-orbit.

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.
- μ_1 generates the G_1-action and μ_2 generates the G_2-action.
- Claim : μ_2 is constant on a G_1-orbit, and similarly μ_1 is constant on a G_2-orbit.
- Proof : $\mu((g, 1)m) = (\mathrm{Ad}_g^* \mu_1(m), \mu_2(m))$,

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.
- μ_1 generates the G_1-action and μ_2 generates the G_2-action.
- Claim : μ_2 is constant on a G_1-orbit, and similarly μ_1 is constant on a G_2-orbit.
- Proof : $\mu((g,1)m) = (\mathrm{Ad}_g^* \mu_1(m), \mu_2(m))$, because $\mathrm{Ad}_{(g,1)}(\xi_1, \xi_2) = (\mathrm{Ad}_g \xi_1, \xi_2)$. □

Reduction for product groups

- Suppose $G := G_1 \times G_2$ be a product of Lie groups, and suppose (M, ω, G, μ) is a Hamiltonian G-space.
- We may write the moment map as $\mu : M \to \mathfrak{g}_1^\vee \oplus \mathfrak{g}_2^\vee, \quad \mu = (\mu_1, \mu_2)$.
- μ_1 generates the G_1-action and μ_2 generates the G_2-action.
- Claim : μ_2 is constant on a G_1-orbit, and similarly μ_1 is constant on a G_2-orbit.
- Proof : $\mu((g,1)m) = (\mathrm{Ad}_g^* \mu_1(m), \mu_2(m))$, because $\mathrm{Ad}_{(g,1)}(\xi_1, \xi_2) = (\mathrm{Ad}_g \xi_1, \xi_2)$. □

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.
- The G_1-action descends to the quotient $\mu_2^{-1}(0)/G_2$, because

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.
- The G_1-action descends to the quotient $\mu_2^{-1}(0)/G_2$, because $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$.

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.
- The G_1-action descends to the quotient $\mu_2^{-1}(0)/G_2$, because $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$.
- We may take a G_1-symplectic quotient of $\mu_2^{-1}(0)/G_2$.

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.
- The G_1-action descends to the quotient $\mu_2^{-1}(0)/G_2$, because $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$.
- We may take a G_1-symplectic quotient of $\mu_2^{-1}(0)/G_2$.
- μ_1 descends to a moment map on $\mu_2^{-1}(0)/G_2$, because μ_1 is G_2-invariant.

Reduction for product groups

- $\mu_2^{-1}(0)$ is G_1-invariant.
- The G_1-action descends to the quotient $\mu_2^{-1}(0)/G_2$, because $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$.
- We may take a G_1-symplectic quotient of $\mu_2^{-1}(0)/G_2$.
- μ_1 descends to a moment map on $\mu_2^{-1}(0)/G_2$, because μ_1 is G_2-invariant.
- In general, 'reduction in stages' can be performed starting with a normal subgroup $H \subset G$.

Example of a product group action

- Last time we saw that the moment map for the $(S^1)^n$-action on $\mathbb{C}^n$

$$(z_1, \ldots, z_n) \xmapsto{(\theta_1, \ldots, \theta_n)} (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

is

$$\mu : \mathbb{C}^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2).$$

Example of a product group action

- Last time we saw that the moment map for the $(S^1)^n$-action on $\mathbb{C}^n$

$$(z_1, \ldots, z_n) \xmapsto{(\theta_1,\ldots,\theta_n)} (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

  is

$$\mu : \mathbb{C}^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2).$$

- Remark : Viewing $(S^1)^n$ as a product of the first factor $G_1 := S^1$ and the rest $G_2 := (S^1)^{n-1}$,

Example of a product group action

- Last time we saw that the moment map for the $(S^1)^n$-action on $\mathbb{C}^n$

$$(z_1, \ldots, z_n) \xmapsto{(\theta_1, \ldots, \theta_n)} (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

  is

$$\mu : \mathbb{C}^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2).$$

- Remark : Viewing $(S^1)^n$ as a product of the first factor $G_1 := S^1$ and the rest $G_2 := (S^1)^{n-1}$, the quotient by G_1 at the level $|z_1|^2 = 1$ is *point* $\times \mathbb{C}^{n-1}$, which has a G_2-action.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi)\rangle = d\langle \phi^* \circ \mu_T, \xi\rangle$.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi) \rangle = d\langle \phi^* \circ \mu_T, \xi \rangle$.
- Equivariance : exercise. □

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi)\rangle = d\langle \phi^* \circ \mu_T, \xi\rangle$.
- Equivariance : exercise. □
- Observe : If $i : G \to T$ is an inclusion, then i^* is restriction.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi) \rangle = d\langle \phi^* \circ \mu_T, \xi \rangle$.
- Equivariance : exercise. □
- Observe : If $i : G \to T$ is an inclusion, then i^* is restriction.
- Example : We recover the moment map for the action of S^1 on $\mathbb{C}^n$ from the action of $(S^1)^n$ on $\mathbb{C}^n$.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi)\rangle = d\langle \phi^* \circ \mu_T, \xi\rangle$.
- Equivariance : exercise. $\square$
- Observe : If $i : G \to T$ is an inclusion, then i^* is restriction.
- Example : We recover the moment map for the action of S^1 on $\mathbb{C}^n$ from the action of $(S^1)^n$ on $\mathbb{C}^n$.
- Here $i : \mathbb{R} \to \mathbb{R}^n$ is $\xi \mapsto (\xi, \dots, \xi)$.

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi)\rangle = d\langle \phi^* \circ \mu_T, \xi\rangle$.
- Equivariance : exercise. □
- Observe : If $i : G \to T$ is an inclusion, then i^* is restriction.
- Example : We recover the moment map for the action of S^1 on $\mathbb{C}^n$ from the action of $(S^1)^n$ on $\mathbb{C}^n$.
- Here $i : \mathbb{R} \to \mathbb{R}^n$ is $\xi \mapsto (\xi, \ldots, \xi)$. The dual $i^* : \mathbb{R}^n \to \mathbb{R}$ is

Actions via group homomorphisms

- Suppose (M, ω, T, μ_T) is a Hamiltonian T-space, and $\phi : G \to T$ is a group homomorphism.
- The group homomorphism induces an action

$$G \times M \to M, \quad g \cdot m := \phi(g)m.$$

- Claim : The moment map of the action is $\mu_G := \phi^* \circ \mu_T$. Here ϕ induces a map $\phi : \mathfrak{g} \to \mathfrak{t}$, whose dual is $\phi^* : \mathfrak{t}^\vee \to \mathfrak{g}^\vee$.
- Proof : $i_{\phi(\xi)_M}\omega = d\langle \mu_T, \phi(\xi) \rangle = d\langle \phi^* \circ \mu_T, \xi \rangle$.
- Equivariance : exercise. □
- Observe : If $i : G \to T$ is an inclusion, then i^* is restriction.
- Example : We recover the moment map for the action of S^1 on $\mathbb{C}^n$ from the action of $(S^1)^n$ on $\mathbb{C}^n$.
- Here $i : \mathbb{R} \to \mathbb{R}^n$ is $\xi \mapsto (\xi, \dots, \xi)$. The dual $i^* : \mathbb{R}^n \to \mathbb{R}$ is $(\xi_1, \dots, \xi_n) \mapsto \sum_i \xi_i$.

Reduction in stages : example

- Now view $G = (S^1)^n$ as the product of

$$H = S^1 \simeq \{(\theta, \dots, \theta) \in (S^1)^n\},$$

$$K = (S^1)^{n-1} \simeq \{(1, \theta_1, \dots, \theta_{n-1}) \in (S^1)^n\}.$$

- The G-action on $\mathbb{C}^n$ has moment map $(z_1, \dots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$.

Reduction in stages : example

- Now view $G = (S^1)^n$ as the product of

$$H = S^1 \simeq \{(\theta, \ldots, \theta) \in (S^1)^n\},$$

$$K = (S^1)^{n-1} \simeq \{(1, \theta_1, \ldots, \theta_{n-1}) \in (S^1)^n\}.$$

- The G-action on $\mathbb{C}^n$ has moment map $(z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.
- The moment map μ_H for the H-action is $(z_1, \ldots, z_n) \mapsto \frac{1}{2}\sum_i |z_i|^2$.

Reduction in stages : example

- Now view $G = (S^1)^n$ as the product of

$$H = S^1 \simeq \{(\theta, \ldots, \theta) \in (S^1)^n\},$$

$$K = (S^1)^{n-1} \simeq \{(1, \theta_1, \ldots, \theta_{n-1}) \in (S^1)^n\}.$$

- The G-action on $\mathbb{C}^n$ has moment map $(z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.
- The moment map μ_H for the H-action is $(z_1, \ldots, z_n) \mapsto \frac{1}{2}\sum_i |z_i|^2$.
- The quotient $\{\mu_H = \frac{1}{2}\}/S^1$ is $\mathbb{P}^{n-1}$.

Reduction in stages : example

- Now view $G = (S^1)^n$ as the product of

$$H = S^1 \simeq \{(\theta, \ldots, \theta) \in (S^1)^n\},$$

$$K = (S^1)^{n-1} \simeq \{(1, \theta_1, \ldots, \theta_{n-1}) \in (S^1)^n\}.$$

- The G-action on $\mathbb{C}^n$ has moment map $(z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.
- The moment map μ_H for the H-action is $(z_1, \ldots, z_n) \mapsto \frac{1}{2} \sum_i |z_i|^2$.
- The quotient $\{\mu_H = \frac{1}{2}\}/S^1$ is $\mathbb{P}^{n-1}$.
- The $(S^1)^{n-1}$-action on $\mathbb{P}^{n-1}$ is $(\theta_1, \ldots, \theta_{n-1}).[z_0 : z_1 : \cdots : z_{n-1}] = [z_0 : e^{i\theta_1} z_1 : \cdots : e^{i\theta_{n-1}} z_{n-1}]$.
- The moment map is the descent of $\mu_K|\{\mu_H = \frac{1}{2}\}$ to the quotient $\{\mu_H = \frac{1}{2}\}/S^1$, which is

Reduction in stages : example

- Now view $G = (S^1)^n$ as the product of

$$H = S^1 \simeq \{(\theta, \ldots, \theta) \in (S^1)^n\},$$

$$K = (S^1)^{n-1} \simeq \{(1, \theta_1, \ldots, \theta_{n-1}) \in (S^1)^n\}.$$

- The G-action on $\mathbb{C}^n$ has moment map $(z_1, \ldots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.
- The moment map μ_H for the H-action is $(z_1, \ldots, z_n) \mapsto \frac{1}{2}\sum_i |z_i|^2$.
- The quotient $\{\mu_H = \frac{1}{2}\}/S^1$ is $\mathbb{P}^{n-1}$.
- The $(S^1)^{n-1}$-action on $\mathbb{P}^{n-1}$ is $(\theta_1, \ldots, \theta_{n-1}).[z_0 : z_1 : \cdots : z_{n-1}] = [z_0 : e^{i\theta_1}z_1 : \cdots : e^{i\theta_{n-1}}z_{n-1}]$.
- The moment map is the descent of $\mu_K|\{\mu_H = \frac{1}{2}\}$ to the quotient $\{\mu_H = \frac{1}{2}\}/S^1$, which is

$$[z_0 : \cdots : z_{n-1}] \mapsto \frac{1}{\sum_{i=1}^{n-1} |z_i|^2}(|z_1|^2, \ldots, |z_{n-1}|^2).$$

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is
$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is
$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$
- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is
$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$
- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices
$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices

$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.

- Claim : μ is a moment map for the T-action.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices

$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.

- Claim : μ is a moment map for the T-action.
- In other words, $\mu_T := i^* \circ \mu$ is a moment map for the T-action on $\mathbb{C}^n$.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is
$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \tfrac{i}{2} z z^*.$$
- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n\times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices
$$T = \{A \in U(n) : A \text{ is diagonal}\}$$
.
- Claim : μ is a moment map for the T-action.
- In other words, $\mu_T := i^* \circ \mu$ is a moment map for the T-action on $\mathbb{C}^n$.
- Here i^* is the dual of the inclusion $i : \mathfrak{t} \to \mathfrak{u}(n)$,

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is
$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$
- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices
$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.
- Claim : μ is a moment map for the T-action.
- In other words, $\mu_T := i^* \circ \mu$ is a moment map for the T-action on $\mathbb{C}^n$.
- Here i^* is the dual of the inclusion $i : \mathfrak{t} \to \mathfrak{u}(n)$, so is equal to the projection of $\mathfrak{u}(n)$ to its diagonal entries.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n \times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices

$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.

- Claim : μ is a moment map for the T-action.
- In other words, $\mu_T := i^* \circ \mu$ is a moment map for the T-action on $\mathbb{C}^n$.
- Here i^* is the dual of the inclusion $i : \mathfrak{t} \to \mathfrak{u}(n)$, so is equal to the projection of $\mathfrak{u}(n)$ to its diagonal entries.
- We have $\mu_T : \mathbb{C}^n \to i\mathbb{R}^n$ is $z \mapsto \frac{i}{2}\,\mathrm{diag}(|z_1|^2, \ldots, |z_n|^2)$.

A moment map for a non-Abelian group

- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} zz^*.$$

- $\mathfrak{u}(n) = \{A \in \mathbb{R}^{n\times n} : A + A^* = 0\}$.
- Identify $\mathfrak{u}(n)^\vee$ to $\mathfrak{u}(n)$ via the Ad-invariant inner product $(A, B) = \mathrm{trace}(A^* B)$.
- Consider the subset of diagonal matrices

$$T = \{A \in U(n) : A \text{ is diagonal}\}$$

.

- Claim : μ is a moment map for the T-action.
- In other words, $\mu_T := i^* \circ \mu$ is a moment map for the T-action on $\mathbb{C}^n$.
- Here i^* is the dual of the inclusion $i : \mathfrak{t} \to \mathfrak{u}(n)$, so is equal to the projection of $\mathfrak{u}(n)$ to its diagonal entries.
- We have $\mu_T : \mathbb{C}^n \to i\mathbb{R}^n$ is $z \mapsto \frac{i}{2}\,\mathrm{diag}(|z_1|^2, \ldots, |z_n|^2)$. The Claim is thus true.

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.
- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- Proof of result : μ is $U(n)$-equivariant.

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.
- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- Proof of result : μ is $U(n)$-equivariant.
- $\mathrm{Ad}_{U(n)}\, \mathfrak{t} = \mathfrak{u}(n)$.

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.
- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} zz^*.$$

- Proof of result : μ is $U(n)$-equivariant.
- $\mathrm{Ad}_{U(n)}\,\mathfrak{t} = \mathfrak{u}(n)$.
- For any $\xi \in \mathfrak{u}(n)$, and $g \in U(n)$,
  $d\mu_\xi = -i_{\xi_M}\omega$ iff $L_g^* d\mu_\xi = -L_g^*(i_{\xi_M}\omega)$

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.
- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} zz^*.$$

- Proof of result : μ is $U(n)$-equivariant.
- $\mathrm{Ad}_{U(n)}\,\mathfrak{t} = \mathfrak{u}(n)$.
- For any $\xi \in \mathfrak{u}(n)$, and $g \in U(n)$,
  $d\mu_\xi = -i_{\xi_M}\omega$ iff $L_g^* d\mu_\xi = -L_g^*(i_{\xi_M}\omega)$
  iff $d\mu_{\mathrm{Ad}_{g^{-1}}\xi} = i_{dL_g^{-1}(\xi_M)}\omega$

A moment map for a non-Abelian group

- A restatement of the Claim : The equation $d\mu_\xi = -i_{\xi_M}\omega$ holds if $\xi \in \mathfrak{t}^\vee$.
- Result : The standard action of $U(n)$ on $(\mathbb{C}^n, \omega_{std})$ is

$$\mu : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- Proof of result : μ is $U(n)$-equivariant.
- $\mathrm{Ad}_{U(n)}\,\mathfrak{t} = \mathfrak{u}(n)$.
- For any $\xi \in \mathfrak{u}(n)$, and $g \in U(n)$,
  $d\mu_\xi = -i_{\xi_M}\omega$ iff $L_g^* d\mu_\xi = -L_g^*(i_{\xi_M}\omega)$
  iff $d\mu_{\mathrm{Ad}_{g^{-1}}\xi} = i_{dL_g^{-1}(\xi_M)}\omega$
  iff the moment equation holds for $\mathrm{Ad}_{g^{-1}}\xi$.
- The result follows because the *Ad*-orbit of any $\xi \in \mathfrak{u}(n)$ intersects $\mathfrak{t}$.

A moment map for a non-Abelian group

- Result : The standard action of $U(k)$ on $(\mathbb{C}^k, \omega_{std})$ is

$$\mu : \mathbb{C}^k \to \mathfrak{u}(k)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

A moment map for a non-Abelian group

- Result : The standard action of $U(k)$ on $(\mathbb{C}^k, \omega_{std})$ is

$$\mu : \mathbb{C}^k \to \mathfrak{u}(k)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- Result : Let $k \leq n$. The standard action of $U(k)$ on $(\mathbb{C}^{k \times n}, \omega_{std})$ is

$$\mu : \mathbb{C}^{k \times n} \to \mathfrak{u}(k)^\vee, \quad A \mapsto \frac{i}{2} A A^*.$$

A moment map for a non-Abelian group

- Result : The standard action of $U(k)$ on $(\mathbb{C}^k, \omega_{std})$ is

$$\mu : \mathbb{C}^k \to \mathfrak{u}(k)^\vee, \quad z \mapsto \tfrac{i}{2} z z^*.$$

- Result : Let $k \leq n$. The standard action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_{std})$ is

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^\vee, \quad A \mapsto \tfrac{i}{2} A A^*.$$

- We may translate by an element in the center of $\mathfrak{u}(n)$ to get

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^\vee, \quad A \mapsto \tfrac{i}{2}(A A^* - \mathrm{Id}).$$

A moment map for a non-Abelian group

- Result : The standard action of $U(k)$ on $(\mathbb{C}^k, \omega_{std})$ is

$$\mu : \mathbb{C}^k \to \mathfrak{u}(k)^\vee, \quad z \mapsto \frac{i}{2} z z^*.$$

- Result : Let $k \leq n$. The standard action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_{std})$ is

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^\vee, \quad A \mapsto \frac{i}{2} A A^*.$$

- We may translate by an element in the center of $\mathfrak{u}(n)$ to get

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^\vee, \quad A \mapsto \frac{i}{2}(AA^* - \mathrm{Id}).$$

- $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}$, and the quotient

$$\{A \in \mathbb{C}^{k\times n} : AA^* = \mathrm{Id}\}/U(k)$$

is the Grassmanian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$.