Introduction to Symplectic Geometry : Lecture 16

October 11, 2021

Moment map

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$$\mu : M \to \mathfrak{g}^{\vee}$$
satisfying
  1. $d\mu_{\xi} = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_{\xi} := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
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- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.

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- Therefore, $\xi_M(m) = 0$ iff $e^{t\xi}m = m$ for all t.

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  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
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- For the orbit $[m] \in M/G$,

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- $\overline{\omega}$ is closed because $\pi^*\overline{\omega}$ is closed, and $\pi^* : \Omega^3(\overline{M}) \to \Omega^3(\mu^{-1}(0))$ is injective.

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- Noether's theorem implies that one can study the dynamics of v_H on the symplectic quotient $\mu^{-1}(c)/G$.

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