Introduction to Symplectic Geometry : Lecture 16

October 11, 2021

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map

$$\mu : M \to \mathfrak{g}^\vee$$

satisfying

1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

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  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.

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- Proof : For any $x \in \mu^{-1}(0)$, $d\mu_x : T_xM \to \mathfrak{g}^\vee$ is onto iff $\langle d\mu_x, \xi\rangle = d(\langle \mu, \xi\rangle)_x \neq 0$ for all $\xi \in \mathfrak{g}$, $\xi \neq 0$.

$d\mu_x$ is onto $\iff$ $\forall\, \xi \in \mathfrak{g}$, $\xi \neq 0$: $\langle d\mu_x, \xi \rangle \neq 0$

$\iff$ $(d\mu_\xi)_x = d\langle \mu, \xi\rangle_x \neq 0 \quad \forall\, \xi \neq 0$

$= -i_{\xi_M}\omega|_x \iff \xi_M(x) \neq 0$

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  Now $d\mu_\xi(x) = \omega(\xi_M(x), \cdot) \neq 0$ because $\xi_M(x) \neq 0$ (by free G-action and the following claim). □

We will prove :
G acts freely ⇒ $\forall \xi \in \mathfrak{g}$ $\xi \neq 0$ $\xi_M \in Vect(M)$ is non-vanishing

$\xi_M(x) := \frac{d}{dt} e^{t\xi} x \big|_{t=0}$

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- Claim : For any $m \in M$, $\xi \in \mathfrak{g}$, $t \mapsto e^{t\xi}m$ is an integral curve of the vector field ξ_M.

Proof Need to check: $\frac{d}{ds} e^{s\xi} m \Big|_{s=t} = \xi_M(e^{\xi t} m)$

$\mathbb{R} \to G$, $t \mapsto e^{t\xi}$ is a homo

$e^{s\xi} = e^{h\xi} e^{t\xi}$

$s = t + h$

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G acts freely $\Rightarrow$ $\forall \xi \neq 0$ $\quad \xi_M(x) \neq 0$

$t \mapsto e^{t\xi} m$ is an int curve

$$\xi_M(m) = 0 \quad \Rightarrow t \mapsto e^{t\xi} m \text{ is constt}$$

$$e^{t\xi} m = m \quad \forall t$$

Contradicts free action

For an arb curve $\mathbb{R} \ni t \mapsto c(t)$ we may have c is non constt $c'(0) = 0$ } Can't happen if c is an integral curve.

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  $= \xi_M(e^{t\xi}m)$. □
- Therefore, $\xi_M(m) = 0$ iff $e^{t\xi}m = m$ for all t.

Symplectic Quotients

Symplectic reduction

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
  - There is a symplectic form $\overline{\omega}$ on $\overline{M}$ satisfying

$$i^*\omega = \pi^*\overline{\omega}$$

where $i : \mu^{-1}(0) \to M$ is the inclusion map.

$\mu^{-1}(0) \xrightarrow{i} M$, $\pi \downarrow$, $\overline{\omega}$ $\overline{M}$

$\mu^{-1}(0)/G$: symp quotient.

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- Example : Consider the diagonal action

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \dots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \dots, z_n)$$

with moment map $\mu(z_1, \dots, z_n) = \frac{1}{2} \sum_i |z_i|^2 - \frac{1}{2}$.

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is compact.

$\Rightarrow$ If G is compact then the G-action is proper

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Theorem

Suppose a Lie group G acts on a manifold M freely and properly. Then

- *the orbit space M/G is a manifold,*
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$$T_m\mathcal{O}_m = \{\xi_M(m) : \xi \in \mathfrak{g}\}. \quad \simeq \operatorname{Ker} d\pi_m$$

$\pi : M \to M/G$

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- For the orbit $[m] \in M/G$,

$$T_{[m]}M/G = T_mM/T_m\mathcal{O}_m.$$

Proof of Marsden-Weinstein-Meyer

0 is a regular value $\Rightarrow \mu^{-1}(0)$ is a mfd.

By the quotient thm

- Since G is a compact group acting freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ is a manifold. For $p \in \mu^{-1}(0)$,

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Need to define symp form on $T_{[p]}(\mu^{-1}(0)/G)$

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  - $T_p\mathcal{O}_p \subset T_p\mu^{-1}(0)$, and

$\mathcal{O}_p \subseteq \mu^{-1}(0)$

since

$\mu^{-1}(0)$ is G-invariant

$T_p\mathcal{O}_p \subseteq T_p\mu^{-1}(0) \subseteq T_pM$

ω_p

Isotropic

Coisotropic

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  - $(T_p\mathcal{O}_p)^\omega = T_p\mu^{-1}(0)$.

$(T_p\mathcal{O}_p)^\omega$
$= \{v : \omega(\xi_M, v) = 0 \quad \forall \xi\}$ → $i_{\xi_M}\omega = -d\mu_\xi$

$= \{v : d\mu_\xi(v) = 0 \quad \forall \xi\}$

$\langle d\mu(v), \xi \rangle = 0 \quad \forall \xi \in \mathfrak{g}$

$d\mu : T_pM \to \mathfrak{g}^\vee$

$(T_p \theta_p)^{\omega}$

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$= \operatorname{Ker} d\mu_p$

$= T_p \mu^{-1}(0)$

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  So, $\mathcal{O}_p$ is isotropic and $\mu^{-1}(0)$ is co-isotropic.

$T_p\mathcal{O}_p \subseteq (T_p\mathcal{O}_p)^\omega$

$T_p\mathcal{O}_p$ is isotropic

In a symp vector space (V, ω)
$I \subseteq V$ $\quad I \subseteq I^\omega$, then I^ω is coisotropic

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- Lemma : Let (V, ω) be a symplectic vector space, and $I \subset V$ be isotropic. Then ω descends to a symplectic form $\overline{\omega}$ on I^ω/I,

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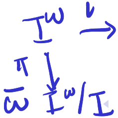

$\pi^*\overline{\omega} = i^*\omega$

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- Proof : Check that $\overline{\omega}$ is well-defined and non-degenerate.

$$I^{\omega} \longrightarrow V$$
$$\downarrow$$
$$\overline{\omega} \quad I^{\omega}/I$$

$v, w \in I^{\omega}$

$\omega(v, w) = 0 \ \forall w \Rightarrow v \in I$

$\because (I^{\omega})^{\omega} = I$

so, $\overline{\omega}$ is non-deg.

$v, w \in I$

$\overline{\omega}(v + I, w + I) := \omega(v, w)$

indep of choice of v, w $\because \omega|_I = 0$

So $\overline{\omega}$ is well-def

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- Proof : Check that $\overline{\omega}$ is well-defined and non-degenerate.
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- Proof : Check that $\overline{\omega}$ is well-defined and non-degenerate.
- For $m \in \mu^{-1}(0)$, apply the lemma taking $I = T_m\mathcal{O}_m$ and so, $I^\omega = T_m\mu^{-1}(0)$, so we get a symplectic linear form $\overline{\omega}_m$ on $T_m\overline{M} = T_m\mu^{-1}(0)/T_m\mathcal{O}_m$

Proof of Marsden-Weinstein-Meyer

- Lemma : Let (V, ω) be a symplectic vector space, and $I \subset V$ be isotropic. Then ω descends to a symplectic form $\overline{\omega}$ on I^ω / I, that satisfies $i^*\omega = \pi^*\overline{\omega}$ where $i : I^\omega \to V$ is the inclusion map and $\pi : I^\omega \to I^\omega / I$ is the projection map.
- Proof : Check that $\overline{\omega}$ is well-defined and non-degenerate.
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- $\overline{\omega}$ is closed because

$\overline{\omega} \in \Omega(\overline{M})$

$\mu^{-1}(0) \xrightarrow{i} M$, $\quad \downarrow \pi$, $\quad \overline{M}$

$i^*\omega = \pi^*\overline{\omega}$

$$\pi^*(d\overline{\omega}) = d(\pi^*\overline{\omega}) = d(i^*\omega) = 0$$

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$

$v, w \in T_m[\mu^{-1}(0)/G]$

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$

$\pi^* \eta = 0$
$\Rightarrow \eta = 0$

π is submersion

$\pi^* : \Omega^3(\mu^{-1}(0)/G) \longrightarrow \Omega^3(\mu^{-1}(0))$

is 1-1

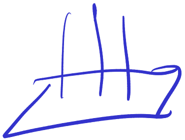
$v_1, v_2, v_3 \in T_{\bar{m}}\, \mu^{-1}(0)/G$

$\pi^*(d\omega)(v_1, v_2, v_3) = 0$

If $d\omega \neq 0$

$\exists\, v_1, v_2, v_3$

$d\omega(v_1, v_2, v_3) \neq 0$



By submersion $\exists$ lifts $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in T\mu^{-1}(0)$

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- $\overline{\omega}$ is closed because $\pi^* \overline{\omega}$ is closed, and $\pi^* : \Omega^3(\overline{M}) \to \Omega^3(\mu^{-1}(0))$ is injective.

Motivation for symplectic reduction : Noether's principle

- Let (M, ω, G, μ) be a Hamiltonian G-space.

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- Let (M, ω, G, μ) be a Hamiltonian G-space. G is connected
- Theorem (Noether) : A Hamiltonian function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the Hamiltonian vector field v_H.

For any $\xi \in \mathfrak{g}$

$$L_{v_H}\mu_\xi = i_{v_H} d\mu_\xi = -i_{v_H} i_{\xi_M}\omega$$

$$= i_{\xi_M} i_{v_H}\omega$$

$$= i_{\xi_M}(dH) = L_{\xi_M} H$$

H is const along ξ_M-orbits $\forall \xi \in \mathfrak{g}$

$\Rightarrow$ H is const on a G-orbit

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- Noether's theorem implies that one can study the dynamics of v_H on the symplectic quotient $\mu^{-1}(c)/G$.

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- μ_1 generates the G_1-action and μ_2 generates the G_2-action. (check)

Claim: μ_2 is const on G_1-orbit

$$(\mu_1, \mu_2)((g_1, 1)m)$$

$$= Ad^*_{(g_1,1)}(\mu_1(m), \mu_2(m))$$

$$= (Ad^*_g \mu_1(m), \mu_2(m))$$

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- 'Reduction in stages' can be performed starting with a normal subgroup $H \subset G$.

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is the dual of the inclusion $i : \mathfrak{h} \to \mathfrak{g}$.

$\xi \in \underline{h}$

$$(i^* \circ \mu)_\xi = \underline{\mu_\xi}$$

$$\langle i^* \circ \mu, \xi \rangle := \langle \mu, \xi \rangle$$

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- Take $G = (S^1)^n$ acting on $\mathbb{C}^n$, and let $H := S^1 = \{(\theta, \dots, \theta)\}$.

$(S^1)^n \circlearrowright \mathbb{C}^n$, $\quad \mu : \mathbb{C}^n \longrightarrow \mathbb{R}^n$

z

$(z_1, \dots z_n) \overset{(\theta_1, \dots \theta_n)}{\longmapsto} (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$

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$\mu : \mathbb{C}^n \to \mathbb{R}^n$, $(z_1, \ldots z_n) \mapsto \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}, \ldots, \frac{|z_n|^2}{2} \right)$

$S^1 \curvearrowright \mathbb{C}^n$ $(z_1 \ldots z_n) \mapsto (e^{i\theta_1} z_1, z_2, \ldots, z_n)$

$$\mu(z_1, \ldots, z_n) = \frac{|z_1|^2}{2}$$

$z_1 = r_1 e^{i\theta_1}$

$\mu : \mathbb{C}^n \to \mathbb{R}$

$$\omega = \sum_{i=1}^{n} r_i dr_i \wedge d\theta_i$$

$d\mu = \underline{r_1 dr_1}$

$\xi_{\mathbb{C}^n} = \partial_{\theta_1}$ $\quad i_{\xi_{\mathbb{C}^n}} \omega = -r_1 dr_1$

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$S^1 \circlearrowright \mathbb{C}^n \qquad (z_1 \dots z_n) \overset{\theta}{\longmapsto} (e^{i\theta} z_1, \dots, e^{i\theta} z_n) \quad (1)$

$S^1 \circlearrowright (S^1)^n \qquad \theta \longmapsto (\theta, \theta, \dots, \theta)$

Moment map for (1) is $\mu(z) = \frac{1}{2} \sum_i |z_i|^2$

$$\begin{array}{ccc} H & & G \\ S^1 & \hookrightarrow & (S^1)^n \end{array} \qquad \theta \mapsto (\theta, \ldots, \theta)$$

$$\mathfrak{h} \xrightarrow{i} \mathfrak{g} \qquad \xi \mapsto (\xi_1, \ldots, \xi)$$

$$i^*: \mathfrak{g}^* \to \mathfrak{h}^* \qquad (\eta_1, \ldots \eta_n) \longmapsto \sum_{i=1}^{n} \eta_i$$

$$\mu_{(S^1)^n}(z) = \left(\frac{|z_1|^2}{2}, \ldots, \frac{|z_n|^2}{2} \right)$$

$$\mu_{(S^1)}(z) = i^* \circ \mu_{(S^1)^n}(z)$$

$$= \sum_i \frac{|z_i|^2}{2}$$

Moment map after reduction at first stage

- Identify G/H with the subtorus

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and has moment map $[z_0 : \cdots : z_{n-1}] \mapsto \frac{1}{\sum_{i=1}^{n-1} |z_i|^2}(|z_1|^2, \dots, |z_{n-1}|^2)$.