Introduction to Symplectic Geometry : Lecture 15

October 6, 2021

Moment map

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satisfying
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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$. Thus μ is an S^1-moment map for the action
$$z \xmapsto{\theta \in S^1} e^{i\theta} z.$$

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- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space.

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- The moment map for the diagonal action is

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- For any $c \in \mathbb{R}$, $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2 + c$ is also a moment map. For general G, we can add central elements $c \in \mathrm{Lie}(Z(G))$ to the moment map without altering the moment map condition.

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- Proof : For any $x \in \mu^{-1}(0), \xi \in \mathfrak{g}\backslash\{0\}$, $d\mu_\xi(x) \neq 0$ because ..

Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
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- Noether's theorem implies that one can study the dynamics of v_H on the symplectic quotient $\mu^{-1}(c)/G$.

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